Sharp Complexity Bounds for Computing Quadrature Formulas for Marginal Distributions of SDEs

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joint work with
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The Problem

Given a family of d-dimensional systems of SDEs

$$\begin{aligned} dX_t &= a(X_t)\,dt + b(X_t)\,dW_t, \quad t \in [0,1], \\ X_0 &= x_0 \in \mathbb{R}^d, \end{aligned}$$

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by a probability measure with finite support

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Note that $\widehat{S}(x_0, a, b)$ yields a quadrature formula

$$\int_{\mathbb{R}^d} f\, d\widehat{S}(x_0, a, b) = \sum_{i=1}^{n} w_i \cdot f(x_i) \quad \text{for} \quad \int_{\mathbb{R}^d} f\, dS(x_0, a, b) = \mathbb{E}\, f(X_1).$$

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- **upper bound:** $\exists\, (A_N)_{N \in \mathbb{N}}$ in $\mathcal{A},\ c > 0\ \forall\, N \in \mathbb{N}$:

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Information-based complexity

Traub, Wasilkowski, Woźniakowski (1988), ...

Algorithms, Error and Cost

Deterministic algorithms $\widehat{S}$

- input: $(x_0, a, b) \in \mathcal{H}$,
- use x_0 and function values of a and b via oracle (subroutine),
- exact computation with real numbers and evaluation of elementary functions,
- output:

$$\widehat{S}(x_0, a, b) = (n, \underbrace{w_1, \dots, w_n}_{\text{weights} \in [0,1]}, \underbrace{x_1, \dots, x_n}_{\text{nodes} \in \mathbb{R}^d}) = \sum_{i=1}^{n} w_i \cdot \delta_{x_i}$$

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Cost of $\widehat{S}$

$$\mathrm{cost}(\widehat{S}, (x_0, a, b)) = \# \text{ evaluations of } a, b + \# \text{ instructions}$$

$$\mathrm{cost}(\widehat{S}) = \sup_{(x_0,a,b) \in \mathcal{H}} \mathrm{cost}(\widehat{S}, (x_0, a, b)).$$

Error of $\widehat{S}$

$$\begin{aligned} \mathrm{error}(\widehat{S}, (x_0, a, b)) &= \rho_F\big(S(x_0, a, b), \widehat{S}(x_0, a, b)\big), \\ \mathrm{error}(\widehat{S}) &= \sup_{(x_0,a,b)\in\mathcal{H}} \mathrm{error}(\widehat{S}, (x_0, a, b)), \end{aligned}$$

where

$$\rho_F(\mu, \widehat{\mu}) = \sup_{f\in F} \left| \int_{\mathbb{R}^d} f \, d\mu - \int_{\mathbb{R}^d} f \, d\widehat{\mu} \right|$$

with a class F of test functions $f : \mathbb{R}^d \to \mathbb{R}$.

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$$\mathrm{error}\big(\widehat{S}, (x_0, a, b)\big) = \rho_F\big(S(x_0, a, b), \widehat{S}(x_0, a, b)\big),$$
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Minimal errors

$$e_N(\mathcal{H}, F) = \inf\{\mathrm{error}\big(\widehat{S}\big) \colon \ \mathrm{cost}\big(\widehat{S}\big) \le N\}.$$

Assumptions and Results

Let $s_1, s_2, r \in \mathbb{N}$ and $\beta \geq 0$.

Class of SDEs $\mathcal{H}_{s_1,s_2} = [-1,1]^d \times H_{s_1}^d \times H_{s_2}^{d\times d}$ with

$$H_s = \{h \in C^s(\mathbb{R}^d;\mathbb{R}) \colon |h(0)|, \|h^{(\alpha)}\|_\infty \leq 1,\ 1 \leq |\alpha| \leq s\}.$$

Metric ρ_{F_r} with

$$F_r = \{f \in C^r(\mathbb{R}^d;\mathbb{R}) \colon |f^{(\alpha)}(x)| \leq 1 + \|x\|^\beta,\ 1 \leq |\alpha| \leq r\}.$$

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Theorem (M-G, Ritter, Yaroslavtseva 2012) For every $\varepsilon > 0$,

$$N^{-\min(s_1,s_2,r)/d} \preceq e_N(\mathcal{H}_{s_1,s_2}, F_r) \preceq N^{-\min(s_1,s_2,r)/d+\varepsilon}.$$

The Lower Bounds

To show:

$$e_N(\mathcal{H}_{s_1,s_2}, F_r) \geq c \cdot \max(N^{-r/d}, N^{-s_1/d}, N^{-s_2/d}).$$

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Use

- $e_N(\mathcal{H}_{s_1,s_2}, F_r) \geq e_N(\mathcal{H}, F)$ if $\mathcal{H} \subset \mathcal{H}_{s_1,s_2}$ and $F \subset F_r$,
- results on minimal errrors for weighted integration on $\mathbb{R}^d$, see Wasilkowski, Woźniakowski (2000, 2001).

Case $d = 1$, lower bound N^{-s_1}, see Petras, Ritter (2006): take

$$\mathcal{H} = \{(0, a, 1) \colon a \in H_{s_1}, \|a\|_\infty \le N^{-s_1}\}, F = \{f\}$$

with $f \in F_r$, $f \ne const$. Then

$$e_N(\mathcal{H}, F) = \inf_{\mathrm{cost}(\widehat{S}) \le N} \sup_a \left| \int_{\mathbb{R}} f \, dS(0, a, 1) - \int_{\mathbb{R}} f \, d\widehat{S}(0, a, 1) \right| \quad .$$

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We have

$$\int_{\mathbb{R}} f \, dS(0, a, 1) = \int_{\mathbb{R}} f(x) \cdot \varphi_a(x) \, dx$$

with a probability density φ_a of $S(0, a, 1)$.

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Series expansion of φ_a by means of the parametrix method yields

$$\int_{\mathbb{R}} f(x) \cdot \varphi_a(x) \, dx = c_f + \int_{\mathbb{R}} a(x) \cdot \underbrace{w_f}_{\neq 0}(x) \, dx + \underbrace{\sum_{m=2}^{\infty} I_m(f, a^{\otimes^m})}_{|\cdot| \leq c_f \cdot N^{-2s_1}} \quad .$$

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The Upper Bounds

Assume $r = s_1 = s_2 = s$ and $d = 1$.

Construct $\widehat{S}_{n,\varepsilon}$ with

$$\mathrm{cost}(\widehat{S}_{n,\varepsilon}) \leq c \cdot n^{1+\varepsilon} \wedge \mathrm{error}(\widehat{S}_{n,\varepsilon}) \leq c \cdot n^{-s}.$$

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Fix n, ε.

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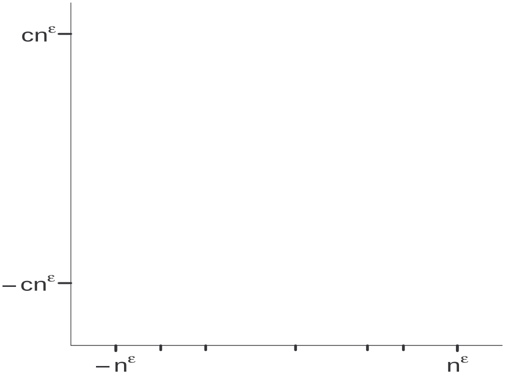

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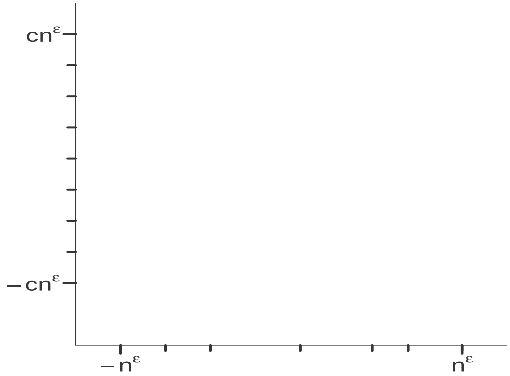

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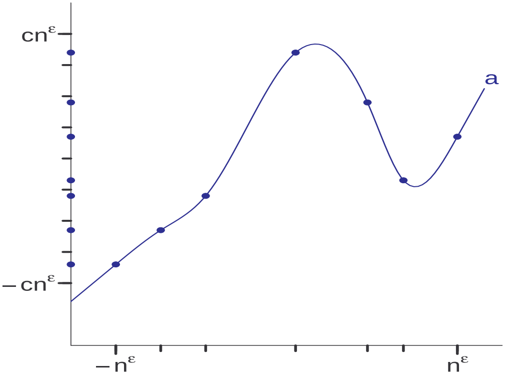

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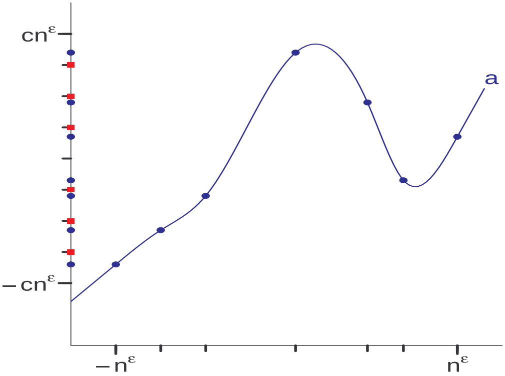

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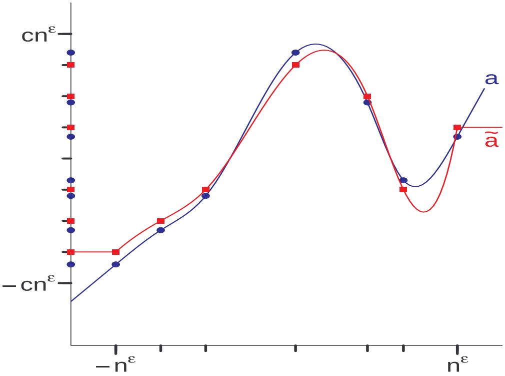

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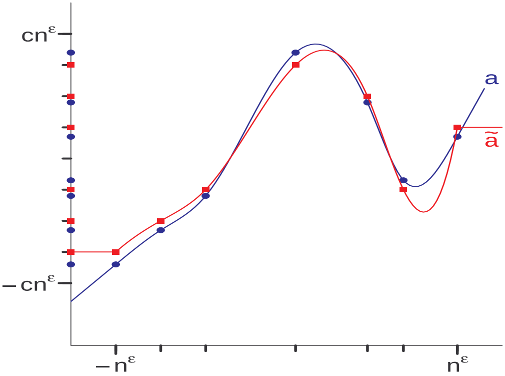

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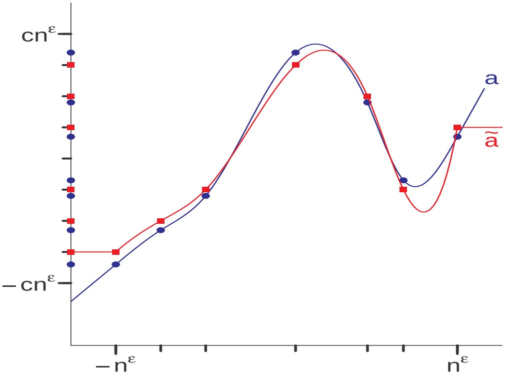

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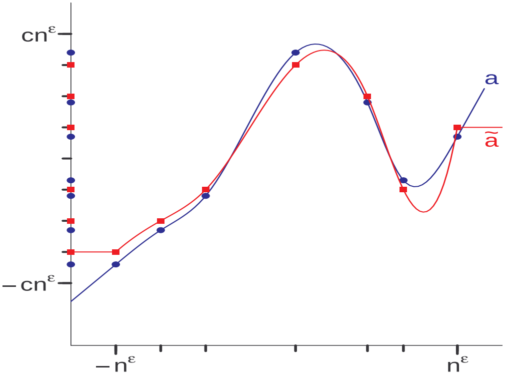

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By a comparison theorem for SDEs:

$$\rho_{F_1}(S(x_0, a, b), S(\widetilde{x}_0, \widetilde{a}, \widetilde{b})) \le c \cdot n^{-s}.$$

(II) Use approximations $\widehat{S}(\widetilde{x}_0, \widetilde{a}, \widetilde{b})$ for $(\widetilde{x}_0, \widetilde{a}, \widetilde{b}) \in \widetilde{\mathcal{H}}$ s.t.

$$|\operatorname{supp}(\widehat{S}(\widetilde{x}_0, \widetilde{a}, \widetilde{b}))| \leq c \cdot n^{1+\varepsilon} \wedge \rho_{F_s}(S(\widetilde{x}_0, \widetilde{a}, \widetilde{b}), \widehat{S}(\widetilde{x}_0, \widetilde{a}, \widetilde{b})) \leq c \cdot n^{-s}.$$

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a) Iterative quantization of the Euler scheme with step-size $\Delta = n^{-2s}$:

$$\widehat{X}_{k+1} = \widehat{X}_k + \widetilde{a}(\widehat{X}_k) \cdot \Delta + \widetilde{b}(\widehat{X}_k) \cdot \Delta_k W, \quad k = 0, \dots, n^{2s},$$

$T \colon \mathbb{R} \to \mathbb{R}$, with appropriate finite range $T(\mathbb{R})$,

$$\overline{X}_{k+1} = \widetilde{X}_k + \widetilde{a}(\widetilde{X}_k) \cdot \Delta + \widetilde{b}(\widetilde{X}_k) \cdot \Delta_k W,$$

$$\widetilde{X}_{k+1} = T(\overline{X}_{k+1}).$$

Yields a measure $Q = P_{\widetilde{X}_{n^{2s}}}$ with

$$|\operatorname{supp}(Q)| \le c \cdot n^{3s} \quad \wedge \quad \rho_{F_1}(S(\widetilde{x}_0, \widetilde{a}, \widetilde{b}), Q) \le c \cdot n^{-s}.$$

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a) Iterative quantization of the Euler scheme with step-size $\Delta = n^{-2s}$:

$$\widehat{X}_{k+1} = \widehat{X}_k + \widetilde{a}(\widehat{X}_k) \cdot \Delta + \widetilde{b}(\widehat{X}_k) \cdot \Delta_k W, \quad k = 0, \dots, n^{2s},$$

$$T\colon \mathbb{R} \to \mathbb{R}, \text{ with appropriate finite range } T(\mathbb{R}),$$

$$\overline{X}_{k+1} = \widetilde{X}_k + \widetilde{a}(\widetilde{X}_k) \cdot \Delta + \widetilde{b}(\widetilde{X}_k) \cdot \Delta_k W,$$

$$\widetilde{X}_{k+1} = T(\overline{X}_{k+1}).$$

Yields a measure $Q = P_{\widetilde{X}_{n^{2s}}}$ with

$$|\operatorname{supp}(Q)| \le c \cdot n^{3s} \quad \wedge \quad \rho_{F_1}(S(\widetilde{x}_0, \widetilde{a}, \widetilde{b}), Q) \le c \cdot n^{-s}.$$

b) Apply a support reduction algorithm to Q, see Davis (1967).

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$$e_N(\mathcal{H}_{s_1,s_2}, F_r) \asymp N^{-\min(r,s_1,s_2)/d+\varepsilon}.$$

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  a) Quadrature formula on the Wiener space with nodes given by bv-paths; exact for multiple Stratonovich integrals up to order m.

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  Given the discrete approximation μ_ℓ of $P_{X_{t_\ell}}$, solve the SDE along the scaled bv-paths on subinterval $[t_\ell, t_{\ell+1}]$ and recombine to get $\mu_{\ell+1}$.

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- Sparse Markov chain based on
  Wagner-Platen scheme with non-uniform time steps

  M-G, Yaroslavtseva (2012)

  Class of SDEs: $d = 1$, $x_0 \in [-1, 1]$, $a, b \in C_b^s$ with $s \geq 6$, $|b| \geq \delta$.
  Metric: ρ_F with $F = \mathrm{Lip}_1(\mathbb{R}; \mathbb{R})$.

  Algorithms $\widehat{S}_n$ with

$$\mathrm{error}(\widehat{S}_n) \preceq \mathrm{cost}(\widehat{S}_n)^{-(1-\varepsilon)}.$$