A Stochastic Galerkin Method for Uncertainty Propagation in Conservation Laws

Olivier Le Maître[1]

[1] LIMSI, CNRS
UPR-3251, Orsay, France
http://www.limsi.fr/Individu/olm

KAUST: 07/11/2012

Simulation framework.

Basic ingredients

- Selection of a mathematical model :
  retain essential physical processes.
- Selection of a numerical method :
  to solve the model equations.
- Define all input-data needed :
  select a specific system in the class spanned by the model.

Simulation errors

- Model errors : physical approximations and simplifications.
- Numerical errors : discretization, approximate solvers, finite arithmetics, ...
- Input-data error : boundary/initial conditions, model constants and parameters, external forcings, ...

Sources of data uncertainty

- Inherent variability (*e.g.* industrial processes).
- Epistemic uncertainty (*e.g.* model constants).
- May not be fully reducible, even theoretically.

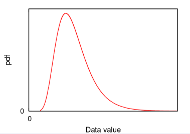
Probabilistic framework

- Define an abstract probability space $(\Theta, \mathcal{A}, d\mu)$.
- Consider input-data D as random quantity : $D(\theta),\ \theta \in \Theta$.
- Simulation output S is random and on $(\Theta, \mathcal{A}, d\mu)$.
- Data D and simulation output S are **dependent** random quantities (through the mathematical model $\mathcal{M}$) :

$$\mathcal{M}(S(\theta), D(\theta)) = 0, \quad \forall \theta \in \Theta.$$

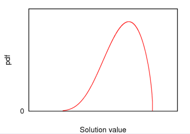
Propagation of data uncertainty

Data density



$$\mathcal{M}(S, D) = 0$$

Solution density



- Variability in model output : numerical error bars.
- Assessment of predictability.
- Support decision making process.
- What type of information (abstract quantities, confidence intervals, density estimations, structure of dependencies, ...) one needs ?

Polynomial Chaos expansions

[Wiener, 1938]

Any well behaved RV $U(\theta)$ (*e.g.* 2nd-order one) defined on $(\Theta, \mathcal{A}, d\mu)$ has a convergent expansion of the form :

$$
\begin{aligned}
U(\theta) &= u_0\Gamma_0 + \sum_{i_1=1}^{\infty} u_{i_1}\Gamma_1(\xi_{i_1}(\theta)) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} u_{i_1,i_2}\Gamma_2(\xi_{i_1}(\theta),\xi_{i_2}(\theta)) \\
&+ \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} u_{i_1,i_2,i_3}\Gamma_3(\xi_{i_1}(\theta),\xi_{i_2}(\theta),\xi_{i_3}(\theta)) + \dots
\end{aligned}
$$

- $\{\xi_1, \xi_2, \dots\}$: independent normalized Gaussian RVs.
- Γ_p polynomials with degree p, orthogonal to $\Gamma_q, \forall q < p$.
- Convergence in the mean square sense [Cameron & Martin, 1947].

Polynomial Chaos expansions

[Wiener, 1938]

Truncated PC expansion at order No and using N RVs :

$$U(\theta) \approx \sum_{k=0}^{\mathrm{P}} u_k \Psi_k(\boldsymbol{\xi}(\theta)), \quad \boldsymbol{\xi} = \{\xi_1, \dots, \xi_{\mathrm{N}}\}, \quad \mathrm{P} = \frac{(\mathrm{N}+\mathrm{No})!}{\mathrm{N}!\mathrm{No}!}.$$

- $\{u_k\}_{k=0,\dots,\mathrm{P}}$: deterministic expansion coefficients,
- $\{\Psi_k\}_{k=0,\dots,\mathrm{P}}$: $\perp$ random polynomials wrt the inner product involving the density of $\boldsymbol{\xi}$:

$$\begin{aligned}\mathbb{E}\{\Psi_k\Psi_l\} = \langle \Psi_k, \Psi_l \rangle \quad &\equiv \int_\Theta \Psi_k(\boldsymbol{\xi}(\theta))\Psi_l(\boldsymbol{\xi}(\theta))d\mu(\theta) \\ &= \int \Psi_k(\boldsymbol{\xi})\Psi_l(\boldsymbol{\xi})p(\boldsymbol{\xi})d\boldsymbol{\xi} = \delta_{kl}\langle \Psi_k, \Psi_k \rangle .\end{aligned}$$

- Gaussian RVs : $p(\boldsymbol{\xi}) = \prod_{i=1}^{\mathrm{N}} \frac{\exp(-\xi_i^2/2)}{\sqrt{2\pi}} \Longrightarrow$ Hermite polynomials (Wiener-Hermite expansions)
- $\{\Psi_0, \dots, \Psi_{\mathrm{P}}\}$ is an orthogonal basis of $\mathcal{S}^{\mathrm{P}} \subset L^2(\mathbb{R}^{\mathrm{N}}, p(\boldsymbol{\xi}))$.

Polynomial Chaos expansions

[Wiener, 1938]

Truncated PC expansion :

$$U(\theta) \approx \sum_{k=0}^{\mathrm{P}} u_k \Psi_k(\xi(\theta)).$$

- Convention $\Psi_0 \equiv 1$: mean mode.
- Expectation of U :

$$\mathbb{E}\left\{U\right\} \equiv \int_{\Theta} U(\theta) d\mu(\theta) \approx \sum_{k=0}^{\mathrm{P}} u_k \int_{\Xi} \Psi_k(\xi) p(\xi) d\xi = u_0.$$

- Variance of U :

$$V[U] = \mathbb{E}\left\{U^2\right\} - \mathbb{E}\left\{U\right\}^2 \approx \sum_{k=1}^{\mathrm{P}} u_k^2 \left\langle \Psi_k, \Psi_k \right\rangle .$$

- Extension to random vectors & stochastic processes :

$$\begin{pmatrix} U_1 \\ \vdots \\ U_m \end{pmatrix} (\theta, \boldsymbol{x}, t) \approx \sum_{k=0}^{\mathrm{P}} \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}_k (\boldsymbol{x}, t) \, \Psi_k(\xi(\theta)).$$

Generalized PC expansion

[Xiu & Karniadakis, 2002]

Askey scheme

Distribution of ξ_i	Polynomial familly
Gaussian	Hermite
Uniform	Legendre
Exponential	Laguerre
β-distribution	Jacobi

Also : discrete RVs (Poisson process).

$$\boxed{U(\theta) \approx \sum_{k=0}^{\mathrm{P}} u_k \Psi_k(\xi(\theta))}$$

where Ψ_k : classical (or product of) polynomials :
spectral expansions

Instead of a spectral expansion over Ξ one can use Piecewise polynomial expansion on a mesh Σ of Ξ

- $\Xi = \bigcup_{SE \in \Sigma} \Xi_{SE}$, $\Xi_{SE} \cap \Xi_{SE'} = \emptyset$ for $SE \neq SE'$
- $\mathcal{S} = \left\{ U \in L^2(\Xi, p_{\xi}), U(\xi \in \Xi_{SE}) \in \mathbb{P}^{\mathrm{N}}_{\mathrm{No}}(\Xi_{SE}) \right\}$

$$\boxed{U(\theta) \approx \sum_{k=0}^{\mathrm{P}} u_k \Psi_k(\xi(\theta))}$$

- Ψ_k are orthogonal with :
  1. Support of Ψ_k limited to an element : Stochastic multi-element method [Deb et al., 2001], [Wang and Karniadakis, 2005]
     Fully decouple the approximation problems over different elements
  2. Hierarchical orthogonal functions Ψ_k : Stochastic Multiwavelet method [olm et al., 2004, 2006, 2009]
     Coupled problems, well suited for adaptive strategy (MRA)

[Ghanem & Spanos, 1991]

Input-data parametrization

Parametrization of D using $\mathrm{N} < \infty$ **independent** RVs with prescribed distribution $p(\xi)$:

$$D(\theta) \approx D(\boldsymbol{\xi}(\theta)), \quad \boldsymbol{\xi} = (\xi_1, \ldots, \xi_{\mathrm{N}}) \in \Xi.$$

- Iso-probabilistic transformations of RVs,
- Karhunen-Loève expansion : $D(\boldsymbol{x}, \theta)$ stochastic field/process,
- Indentification (*e.g.* Bayesian).

Model

Solution expansion

[Ghanem & Spanos, 1991]

Input-data parametrization

Model

We assume that for a.e. $\xi \in \Xi$, the problem $\mathcal{M}(S, D(\xi)) = 0$

1. is well-posed,
2. has a unique solution

and that

$$\boxed{\text{the random solution } S(\xi) \in L^2(\Xi, p_\xi)} :$$

$$\mathbb{E}\left\{S^2\right\} = \int_\Theta S^2(\xi(\theta))d\mu(\theta) = \int_\Xi S^2(\xi)p(\xi)d\xi < +\infty.$$

Solution expansion

[Ghanem & Spanos, 1991]

Input-data parametrization

Model

Solution expansion

Let $\{\Psi_0, \Psi_1, \ldots\}$ be a basis of $L^2(\Xi, p_\xi)$ then

$$S(\xi) = \sum_k s_k \Psi_k(\xi).$$

- Knowledge of the spectral coefficients s_k fully determine the random solution.
- Makes explicit the dependence between $D(\xi)$ and $S(\xi)$.

[Ghanem & Spanos, 1991]

Input-data parametrization

Model

Solution expansion

Let $\{\Psi_0, \Psi_1, \ldots\}$ be a basis of $L^2(\Xi, p_\xi)$ then

$$S(\xi) = \sum_k s_k \Psi_k(\xi).$$

- Knowledge of the spectral coefficients s_k fully determine the random solution.
- Makes explicit the dependence between $D(\xi)$ and $S(\xi)$.
- Need efficient procedure(s) to compute the s_k.

Galerkin projection

Method of weighted residual

① Introduce truncated expansions in model equations

② Require residual to be $\perp$ to the stochastic subspace $\mathcal{S}^{\mathrm{P}}$

$$\left\langle \mathcal{M}\left(\sum_{k=0}^{\mathrm{P}} s_k \Psi_k(\xi), D(\xi)\right), \Psi_m(\xi)\right\rangle = 0 \quad \text{for } m = 0, \ldots, \mathrm{P}.$$

Set of $\mathrm{P}+1$ coupled problems.

Plus

- Implicitly account for modes' coupling
- Often inherit properties of the deterministic model

Minus

- Requires adaptation of deterministic solvers
- Treatment of non-linearities.

PhD work of Julie Tryoen

with Alexandre Ern (Cermics, Univ. Paris-Est)
and Michael Ndjinga (CEA, Saclay)

Hyperbolic systems : deterministic case

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot \boldsymbol{f}(\boldsymbol{u}) = 0, \quad \boldsymbol{u}(\boldsymbol{x}, t=0) = \boldsymbol{u}^0(\boldsymbol{x}), \quad BCs$$

- $\boldsymbol{u} \in \mathcal{A}_{\boldsymbol{u}} \subset \mathbb{R}^m$ (conservative variables)
- $\boldsymbol{f} : \mathcal{A}_{\boldsymbol{u}} \mapsto \mathbb{R}^m$ (flux function)
- if $\nabla_{\boldsymbol{u}} \boldsymbol{f} \in \mathbb{R}^{m \times m}$ is $\mathbb{R}$-diagonalizable on $\mathcal{A}_{\boldsymbol{u}} \Longrightarrow$ hyperbolic
- $\boldsymbol{u}$ can develop shocks / discontinuities in finite time

Classical discretization (Finite Volume in 1-space dimension)

$$\frac{\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_i^n}{\Delta t} + \frac{\widetilde{\boldsymbol{f}}(\boldsymbol{u}_i^n, \boldsymbol{u}_{i+1}^n) - \widetilde{\boldsymbol{f}}(\boldsymbol{u}_{i-1}^n, \boldsymbol{u}_i^n)}{\Delta x} = 0$$

where $\boldsymbol{u}_i^n = \int_{\Delta x} \boldsymbol{u}(x, t_n) \mathrm{d}x$ and $\widetilde{\boldsymbol{f}}(,)$ is the numerical flux function (having had-hoc properties).

Uncertain hyperbolic problems :

- Uncertain initial & boundary conditions and parameters in $\boldsymbol{f}$
- Parametrization with $\xi(\theta) = \{\xi_1(\theta), \dots; \xi_{\mathrm{N}}(\theta)\}$ a set of N iid random variables with uniform distribution on $\Xi = [0,1]^{\mathrm{N}}$
- Stochastic Hyperbolic Problem

$$\frac{\partial \boldsymbol{U}(\boldsymbol{x}, t, \xi)}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{U}; \xi) = 0, \quad \boldsymbol{U}(\boldsymbol{x}, t = 0, \xi) = \boldsymbol{U}^0(\boldsymbol{x}, \xi) \quad (a.s.)$$

Hypotheses

1. $\boldsymbol{U}(\boldsymbol{x}, t, \xi) \in \mathcal{A}_{\boldsymbol{U}}$ and $\nabla_{\boldsymbol{U}} \boldsymbol{F}(\boldsymbol{U}; \xi)$ is $\mathbb{R}$-diagonalizable a.s.
2. all random quantities have finite variance.

The solution is sought in $\mathcal{S}^{\mathrm{P}} := \mathrm{span}\,\{\Psi_0, \dots, \Psi_{\mathrm{P}}\} \subset L_2(\Xi)$, where Ψ_α are orthonormal polynomials in ξ with degree $\leq \mathrm{No}$: $\langle \Psi_\alpha, \Psi_\beta \rangle = \delta_{\alpha,\beta}$.

Galerkin problem :

- Since $\boldsymbol{U} \in L^2(\Xi)$ it has a convergent expansion :

$$\boldsymbol{U}(\boldsymbol{x}, t, \xi) = \sum_{\alpha} \boldsymbol{u}_\alpha(x, t)\Psi_\alpha(\xi)$$

- We denote $\boldsymbol{U}^{\mathrm{P}}$ the approximation of $\boldsymbol{U}$ in $\mathcal{S}^{\mathrm{P}}$
- Stochastic Galerkin projection of the hyperbolic problem : for $\alpha = 0, \dots, \mathrm{P}$

$$\frac{\partial \boldsymbol{u}_\alpha(\boldsymbol{x}, t)}{\partial t} + \nabla \cdot \boldsymbol{f}_\alpha(\boldsymbol{u}_0, \dots, \boldsymbol{u}_{\mathrm{P}}) = 0$$

$$\boldsymbol{f}_\alpha(\boldsymbol{u}_0, \dots, \boldsymbol{u}_{\mathrm{P}}) \equiv \left\langle \boldsymbol{F}(\boldsymbol{U}^{\mathrm{P}}; \xi), \Psi_\alpha \right\rangle$$

$$\boldsymbol{u}_\alpha(\boldsymbol{x}, t = 0) = \left\langle \boldsymbol{U}^0(\boldsymbol{x}), \Psi_\alpha \right\rangle$$

$(\mathrm{P}+1)$-coupled problems for the solution modes

Galerkin problem : (system form)

$$\frac{\partial}{\partial t}\begin{pmatrix} \boldsymbol{u}_0 \\ \vdots \\ \boldsymbol{u}_{\mathrm{P}} \end{pmatrix} + \boldsymbol{\nabla}\cdot\begin{pmatrix} \boldsymbol{f}_0(\boldsymbol{u}_0,\dots,\boldsymbol{u}_{\mathrm{P}}) \\ \vdots \\ \boldsymbol{f}_{\mathrm{P}}(\boldsymbol{u}_0,\dots,\boldsymbol{u}_{\mathrm{P}}) \end{pmatrix} = 0$$

$$\frac{\partial \mathcal{U}}{\partial t} + \boldsymbol{\nabla}\cdot\mathcal{F}(\mathcal{U}) = 0$$

- $\mathcal{U} \in \mathbb{R}^{m\times(\mathrm{P}+1)}$
- $\mathcal{F} : \mathbb{R}^{m\times(\mathrm{P}+1)} \mapsto \mathbb{R}^{m\times(\mathrm{P}+1)}$
- Is the Galerkin problem hyperbolic ?
- ($\nabla_{\mathcal{U}}\mathcal{F}$ $\mathbb{R}$-diagonalizable ?)
- What is the admissible domain $\mathcal{A}_{\mathcal{U}}$?

Jacobian of the Galerkin problem

$$\nabla_{\mathcal{U}}\mathcal{F} = \begin{pmatrix} \mathcal{F}'_{0,0} & \cdots & \mathcal{F}'_{0,\mathrm{P}} \\ \vdots & \ddots & \vdots \\ \mathcal{F}'_{\mathrm{P},0} & \cdots & \mathcal{F}'_{\mathrm{P},\mathrm{P}} \end{pmatrix}, \quad \mathcal{F}'_{\alpha,\beta} = \left\langle \nabla_{\boldsymbol{u}}\boldsymbol{F}(\boldsymbol{U}^{\mathrm{P}};\boldsymbol{\xi}), \Psi_\alpha\Psi_\beta \right\rangle \in \mathbb{R}^{m,m}$$

- If $\nabla_u \boldsymbol{F}$ is symmetric (a.s.), $\nabla_{\mathcal{U}}\mathcal{F}$ is $\mathbb{R}$-diagonalizable
- In particular, scalar problems ($m = 1$) yield hyperbolicity
- If $\nabla_{\boldsymbol{u}}\boldsymbol{F} = \boldsymbol{L}\boldsymbol{D}(\xi)\boldsymbol{R}$, where $\boldsymbol{L}$ and $\boldsymbol{R}$ are deterministic, the Galerkin problem is hyperbolic
- Properties extend to $\neq$ truncature rules
- Note that strict hyperbolicity is **not** to be expected even when $\nabla_{\boldsymbol{u}}\boldsymbol{F}$ has (a.s.) distinct eigenvalues.

[J. Tryoen et al, JCP 2010, JCAM 2010]

General case

Let $\{\xi^{(i)}\}$, and $\{w^{(i)}\}$, $i = 0, \dots, \mathrm{P}$ the points and weights of the Gauss' quadrature on Ξ.
Define

$$\left(\overline{\nabla_{\mathcal{U}}\mathcal{F}}\right)_{\alpha,\beta} = \sum_{i=0}^{\mathrm{P}} \nabla_{\boldsymbol{U}} \boldsymbol{F}\left(\boldsymbol{U}^{\mathrm{P}}(\xi^{(i)}); \xi^{(i)}\right) \Psi_{\alpha}\left(\xi^{(i)}\right) \Psi_{\beta}\left(\xi^{(i)}\right) w^{(i)} \approx \mathcal{F}'_{\alpha,\beta}$$

- $\Rightarrow \overline{\nabla_{\mathcal{U}}\mathcal{F}}$ is $\mathbb{R}$-diagonalizable
- Let $\{\Lambda^l(\xi)\}_{l=1}^{l=m}$, the stochastic Eigenvalues of $\nabla \boldsymbol{F}$
  $\{\Lambda^l_i \equiv \Lambda^l(\xi^{(i)})\}$ **are the eigenvalues of $\overline{\nabla_{\mathcal{U}}\mathcal{F}}$**
- For sufficient smoothness, $\lim_{\mathrm{No}\to\infty} \overline{\nabla_{\mathcal{U}}\mathcal{F}} = \nabla_{\mathcal{U}}\mathcal{F}$
- Use $\{\Lambda^l(\xi^{(i)})\}$ as approximate spectrum of $\nabla_{\mathcal{U}}\mathcal{F}$

[J. Tryoen et al, JCP 2010, JCAM 2010]

Approximate Roe solver

$$\mathcal{U}_i^{n+1} = \mathcal{U}_i^n - \frac{\Delta t}{\Delta x}\left[\phi(\mathcal{U}_i^n, \mathcal{U}_{i+1}^n) - \phi(\mathcal{U}_{i-1}^n, \mathcal{U}_i^n)\right]$$

where the numerical flux Φ is chosen as

$$\phi(\mathcal{U}_L, \mathcal{U}_R) = \frac{1}{2}\left[\mathcal{F}(\mathcal{U}_L) + \mathcal{F}(\mathcal{U}_R)\right] - a\frac{\mathcal{U}_R - \mathcal{U}_L}{2}$$

where $a \in \mathbb{R}^{m(\mathrm{P}+1)\times m(\mathrm{P}+1)}$ is a non-negative upwind matrix

Theorem : It exists a Galerkin Roe state $\mathcal{U}_{L,R}^{\mathrm{Roe}}$ such that
$\nabla \mathcal{F}_{\mathcal{U}}(\mathcal{U}_{L,R}^{\mathrm{Roe}})$ is a Roe matrix for the Galerkin problem

i.e. has properties of consistency and conservativity through shocks.

We will take

$$\phi(\mathcal{U}_L, \mathcal{U}_R) = \frac{1}{2}\left[\mathcal{F}(\mathcal{U}_L) + \mathcal{F}(\mathcal{U}_R)\right] - \left|\nabla_{\mathcal{U}} \mathcal{F}(\mathcal{U}_{L,R}^{\mathrm{Roe}})\right| \frac{\mathcal{U}_R - \mathcal{U}_L}{2}$$

where $|A| = |LDR| = L\,|D|\,R$ for a $\mathbb{R}$-diagonalizable matrix

[J. Tryoen et al, JCP 2010]

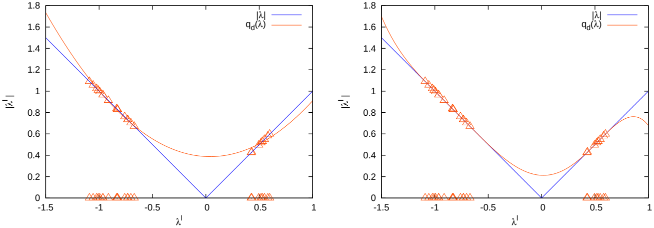
Fast approximation of the upwind matrix

To avoid the costly decomposition of the Roe matrix, we rely on a polynomial transform q_d :

- recall $q(LDR) = Lq(D)R$
- $|\nabla_{\mathcal{U}}\mathcal{F}| \approx q_d\left(\nabla_{\mathcal{U}}\mathcal{F}\right)$, where $q_d \in \mathbb{P}_d$ minimizes

$$J = \sum_{i,l}\left[q_d\left(\Lambda_i^l\right) - \left|\Lambda_i^l\right|\right]^2, \quad \Lambda_i^l \approx \Lambda^l\left(\boldsymbol{U}_{LR}^{\mathrm{Roe}}(\boldsymbol{\xi}^{(i)})\right)$$

- In practice $d \leq 6$ is sufficient



Approximation polynomial q_d for $d = 2$ (left) and $d = 6$ (right).

Summary :

$$\mathcal{U}_i^{n+1} = \mathcal{U}_i^n - \frac{\Delta t}{\Delta x}\left[\phi(\mathcal{U}_i^n, \mathcal{U}_{i+1}^n) - \phi(\mathcal{U}_{i-1}^n, \mathcal{U}_i^n)\right]$$

where

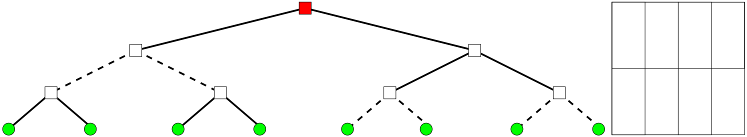
$$\phi(\mathcal{U}_L, \mathcal{U}_R) = \frac{1}{2}\left[\mathcal{F}(\mathcal{U}_L) + \mathcal{F}(\mathcal{U}_R)\right] - q_d\left(\nabla_{\mathcal{U}}\mathcal{F}(\mathcal{U}^{\mathrm{Roe}})\right)\frac{\mathcal{U}_R - \mathcal{U}_L}{2}$$

- Upwinding w.r.t. the actual waves in the Galerkin solution
- Applies conditionally to partially tensored stochastic basis
- May need Entropy corrector [J. Tryoen et al, JCAM 2010]
- Assume $\boldsymbol{U}(\xi)$ smooth and sufficient stochastic discretization
- But solutions are not smooth in general !

Call for piecewise polynomial approximations to allow for discontinuities at the stochastic level

Binary trees for piecewise polynomial space

$\mathcal{S} = \mathcal{S}(\mathrm{T})$



- Dyadic partitions of a node along a prescribe direction $\mathrm{d} : \mathrm{p} \rightarrow (\mathrm{c}^-, \mathrm{c}^+)$
- Piecewise-polynomial with **fixed** order No on each leaf of T.

1. Union of local modal basis : SE-basis

   [Deb et al, 2001], [Karniadakis et al]

   Uncoupled application of the Roe scheme over different leafs

2. Hierarchical global basis over Ξ : MW-Basis

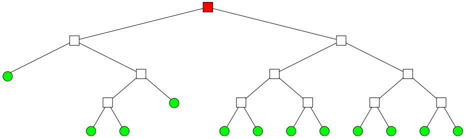
   [OLM et al, 2004]

   Hierarchical sequence of details, suited for adaptive scheme

Adaptivity

Singularity curves are localized in Ξ : **stochastic adaptivity**

- **Incomplete and anisotropic binary trees**



Operators for multi-resolution analysis :

- **Prediction operator** : define the solution in a stochastic space larger than the current one (add new leafs and L^2-injection).
- **Restriction operator** : define the solution in a stochastic space smaller one the current one (remove leafs and L^2-projection).
- Rely on recursive application of **elementary (directional) operators**, full exploitation of the tree structure.

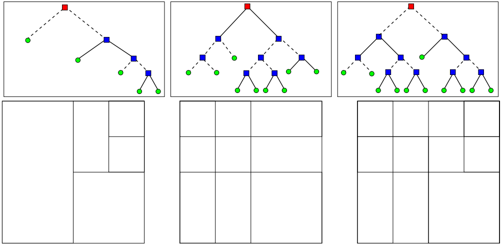
Adaptivity :

Singularity curves are localized in x and t

- **Each spatial cell** carries its own adapted stochastic discretization
- Flux computation,

$$\phi(\mathcal{U}_L, \mathcal{U}_R) = \frac{\mathcal{F}(\mathcal{U}_L) + \mathcal{F}(\mathcal{U}_R)}{2} - \left| a^{\text{Roe}}(\mathcal{U}_L, \mathcal{U}_R) \right| \frac{\mathcal{U}_R - \mathcal{U}_L}{2},$$

  with $\mathcal{U}_R$ and $\mathcal{U}_L$ known on **different stochastic spaces**
- **Union operator** : given two stochastic spaces, construct the minimal stochastic space containing the two :



Adaptive Algorithm :

1. Loop over all interfaces of the spatial mesh :
   - Construct the union space of the left and right cells
   - **Enrich** this space
   - Predict left and right states of the interface
   - Evaluate the numerical flux (App. Roe scheme)
2. Loop over all cells of the spatial mesh :
   - Construct the union space of the cell's interfaces
   - Predict cell's fluxes on the union space
   - Compute fluxes difference and update cell's solution
   - Restrict cell's solution by **thresholding**
3. Repeat for the next time step

Two indicators needed : based on multiwavelet details of nodes.

- for Enrichment : anticipate emergence of new stochastic details,
- for Thresholding : remove unnecessary/negligible details.

Thresholding criterion :

Let us denote

- $\mathtt{T}$ a binary tree and $\mathcal{S}(\mathtt{T})$ the corresponding stochastic approximation space
- $\mathtt{n} \in \mathcal{N}(\mathtt{T})$ a node of the tree, and $\widehat{\mathcal{N}}(\mathtt{T})$ set set of nodes having children
- Nr the maximal depth allowed in a direction
- $\mathtt{T}_{[\mathrm{NNr}]}$ the maximal tree given Nr

We define for $U \in \mathcal{S}(\mathtt{T}_{[\mathrm{NNr}]})$ and $\eta > 0$ the subset of $\mathcal{N}(\mathtt{T}_{[\mathrm{NNr}]})$

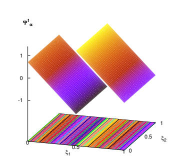
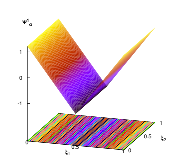
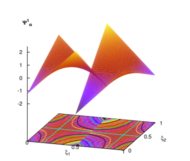
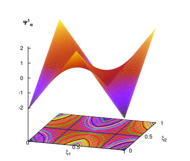
$$\mathcal{D}(\eta) := \left\{ \mathtt{n} \in \widehat{\mathcal{N}}(\mathtt{T}_{[\mathrm{NrN}]}); \|\tilde{\boldsymbol{u}}^{\mathtt{n}}\|_{\ell^2} \leq 2^{-|\mathtt{n}|/2} \frac{\eta}{\sqrt{\mathrm{NNr}}} \right\},$$

where $\tilde{\boldsymbol{u}}^{\mathtt{n}} := (\tilde{u}^{\mathtt{n}}_{\alpha})_{1 \leq \alpha \leq \mathrm{P}}$ are the MW coefficients of $\mathtt{n}$.

Coarsening strategy :
Two sisters $(\mathrm{c}^-, \mathrm{c}^+)$ of a parent $\mathrm{p}(\mathrm{c}^-)$ are removed from the discretization if

$$\|\tilde{\boldsymbol{u}}^{\mathrm{p}(\mathrm{c}^-)}\|_{\ell^2} \leq 2^{-|\mathrm{n}|/2} \frac{\eta}{\sqrt{\mathrm{NNr}}}$$

The criterion ensures that $\|U^{\mathrm{T}[\mathrm{NNr}]} - U^{\mathrm{T}[\mathrm{NNr}] \setminus \mathcal{D}}\| \leq \eta$.

$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$
			

Mother wavelets $\tilde{\Psi}^{\mathrm{d}}_{\alpha}$ for $\mathrm{N} = 2$, $\mathrm{No} = 1$ in direction $\mathrm{d} = 1$.

Note : the coarsening is applied to the class of equivalent trees.

Enrichment strategy :

Enrichment is necessary to anticipate emergence of new-stochastic details.

- Isotropic enrichment is not an option for $\mathrm{N} > 2, 3$
- 1-D enrichment criterion : if U is (locally) smooth enough $\tilde{u}_\alpha^{\mathtt{n}}$ of a generic node $\mathtt{n}$ can be bounded as

$$|\tilde{u}_\alpha^{\mathtt{n}}| = \inf_{P \in \mathbb{P}_{\mathrm{No}}[\xi]} |\langle (U - P), \Psi_\alpha^{\mathtt{n}} \rangle| \leq C |S(\mathtt{n})|^{\mathrm{No}+1} \|U\|_{H^{\mathrm{No}+1}(S(\mathtt{n}))},$$

  where $|S(\mathtt{n})| = 2^{-|\mathtt{n}|}$ is the volume of the node. Therefore $\|\tilde{\boldsymbol{u}}^{\mathtt{n}}\|_{\ell^2} \sim 2^{-(\mathrm{No}+1)} \|\tilde{\boldsymbol{u}}^{\mathtt{p}(\mathtt{n})}\|_{\ell^2}$ and a leaf $\mathtt{l}$ is refined if

$$\|\tilde{\boldsymbol{u}}^{\mathtt{p}(\mathtt{l})}\|_{\ell^2} \geq 2^{\mathrm{No}+1} 2^{-|\mathtt{l}|/2} \eta / \sqrt{\mathrm{Nr}} \quad \text{and} \quad |\mathtt{l}| < \mathrm{Nr}.$$

Enrichment strategy :

Extension of to the N-dimensional case :

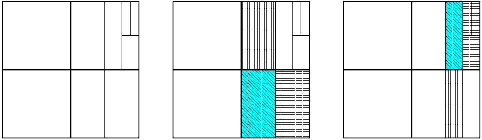
- Using the decay estimation

$$|\tilde{u}^{\mathrm{n}}_{\alpha}| = \inf_{P\in\mathbb{P}^{\mathrm{N}}_{\mathrm{No}}[\xi]} \left|\left\langle (U-P), \Psi^{\mathrm{n},d}_{\alpha}\right\rangle\right| \leq C \mathrm{diam}(S(\mathrm{n}))^{\mathrm{No}+1} \|U\|_{H^{\mathrm{No}+1}(S(\mathrm{n}))},$$

- a leaf $\mathtt{l}$ is partitioned in direction d if

$$\|\tilde{\boldsymbol{u}}^{\mathrm{p}^d(\mathtt{l})}\|_{\ell^2} \geq \left(\frac{\mathrm{diam}(S(\mathrm{p}^d(\mathtt{l})))}{\mathrm{diam}(S(\mathtt{l}))}\right)^{\mathrm{No}+1} 2^{-|\mathtt{l}|/2}\eta/\sqrt{\mathrm{NNr}} \quad \text{and} \quad |S(\mathtt{l})|_d > 2^{-\mathrm{Nr}}.$$

- the construction of the virtual sister and parent of $\mathtt{l}$ in arbitrary direction d



A sharper anisotropic criterion has been proposed using $1-D$ analysis functions in direction d [J.Tryoen, preprint 2012].

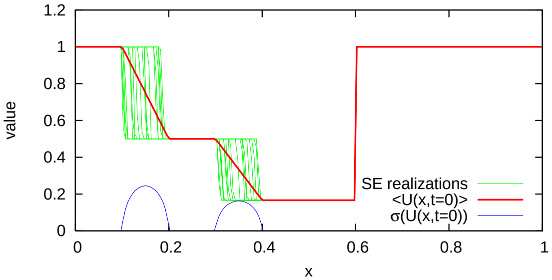
Burgers equation

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2}$$

Uncertain initial condition $U^0(x, \xi)$:

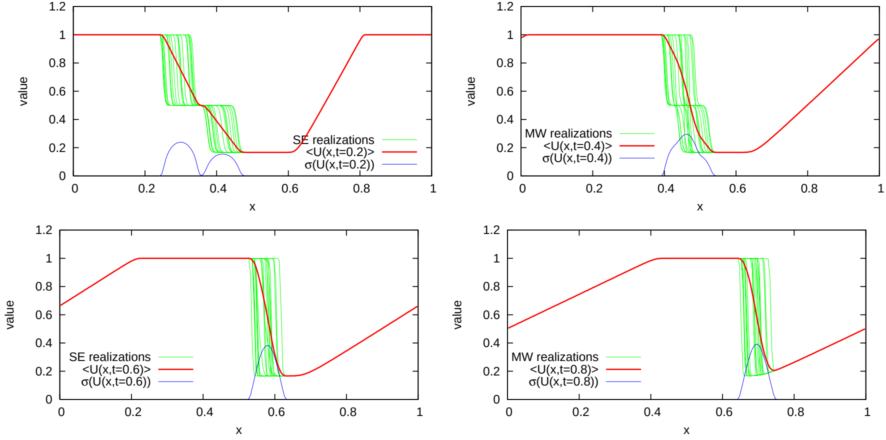
$$X_{1,2} = 0.1 + 0.1\xi_1, \quad X_{2,3} = 0.3 + 0.1\xi_2, \quad \xi_1, \xi_2 \sim \mathcal{U}[0, 1]$$

2 stochastic dimensions.



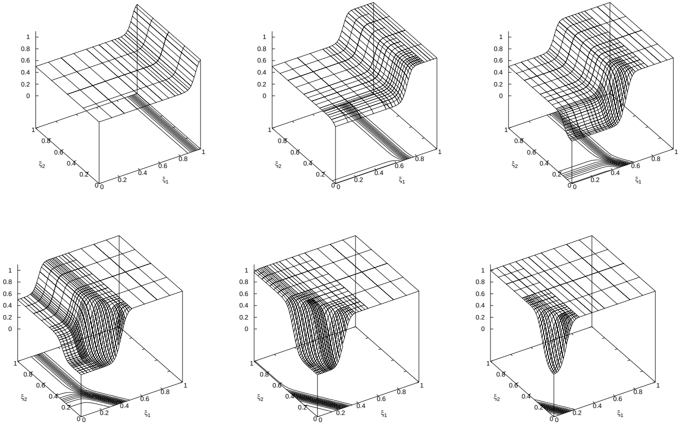
Burgers equation

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2}$$



Burgers equation

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2}$$



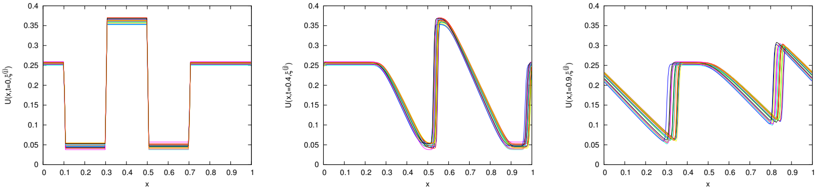
Trafic equation in periodic [0, 1]-domain

$$F(U(\xi);\xi) = A(\xi)U(\xi)(1 - U(\xi)) \quad \text{1-Periodic BC.}$$

- uncertain initial density of vehicles

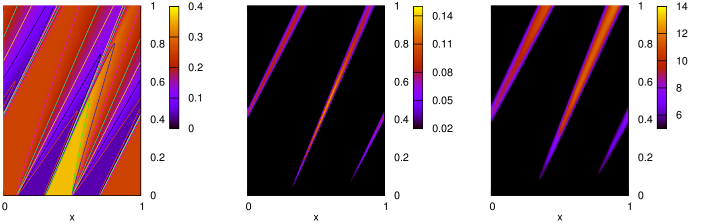
$$\begin{aligned} U^0(x,\xi) =& 0.25 + 0.01\xi_1 - \mathbb{I}_{[0.1,0.3]}(x)(0.2 + 0.015\xi_2) \\ &+ \mathbb{I}_{[0.3,0.5]}(x)(0.1 + 0.015\xi_3) - \mathbb{I}_{[0.5,0.7]}(x)(0.2 + 0.015\xi_4) \end{aligned}$$

- uncertain characteristic velocity $A(\xi) = 1 + 0.1\xi_5$
- 5-dimensional problem $(\xi_1, \ldots, \xi_5) \sim U[0,1]^5$.



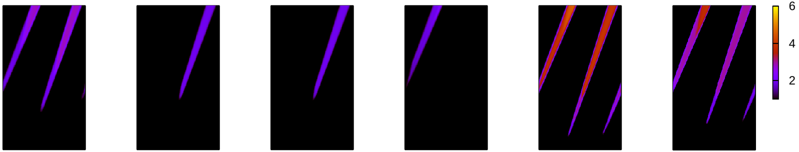
20 realizations of the initial condition (left) and solution at $t = 0.4$ (middle) and $t = 0.9$ (right) : 2 shocks and 2 rarefaction waves.

Space-time diagrams of the solution mean (left), standard deviation (center) and average depth of the leafs (right) :



Averaged number of partitions in each direction D_i and anisotropy factor ρ :

D_1	D_2	D_3	D_4	D_5	ρ



Hoeffding decomposition. Orthogonal hierarchical decomposition

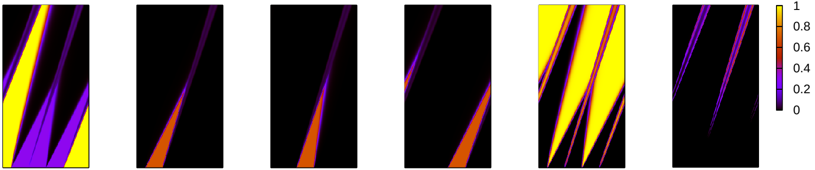
$$U(\xi_1,\ldots,\xi_{\mathrm{N}}) = U_0 + \sum_{i_1=1}^{\mathrm{N}} U_{i_1}(\xi_{i_1}) + \sum_{i_1=1}^{\mathrm{N}} \sum_{i_2=i_1+1}^{\mathrm{N}} U_{i_1,i_2}(\xi_{i_1},\xi_{i_2}) + \ldots$$
$$+ U_{1,\ldots,\mathrm{N}}(\xi_{i_1},\ldots,\xi_{i_{\mathrm{N}}}),$$

Sobol ANOVA (analysis of the variance)

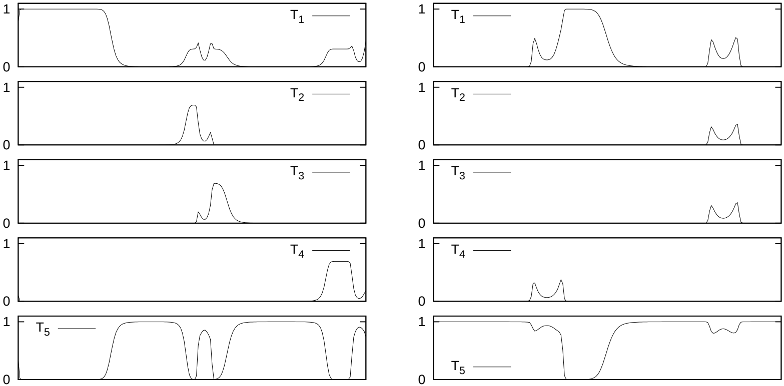
$$V(U) = \sum_{i_1=1}^{\mathrm{N}} V_{i_1} + \sum_{i_1=1}^{\mathrm{N}} \sum_{i_2=i_1+1}^{\mathrm{N}} V_{i_1,i_2} + \cdots + V_{1,\ldots,\mathrm{N}},$$

- First order sensitivity indexes : $\mathrm{S}_i = V_i/V$
- Total sensitivity indexes : $\mathrm{T}_i = \sum_{u\subseteq\{1,\ldots,\mathrm{N}\}}^{u\ni\{i\}} V_u/V$

S_1	S_2	S_3	S_4	S_5	$1 - \sum_d S_d$

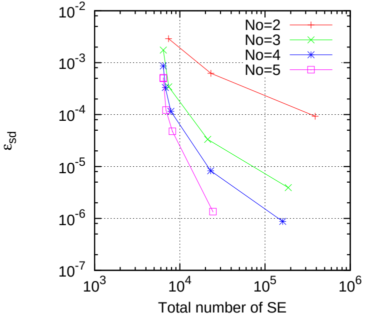


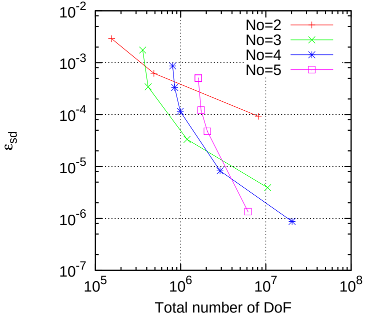
Space-time diagrams of the 1-st order sensitivity indexes S_i and contribution of higher order indexes.



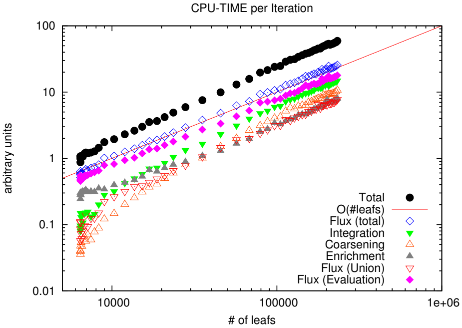
Total sensitivity indices as a function of $x \in [0, 1]$ at $t = 0.4$ (left) and $t = 0.9$ (right).

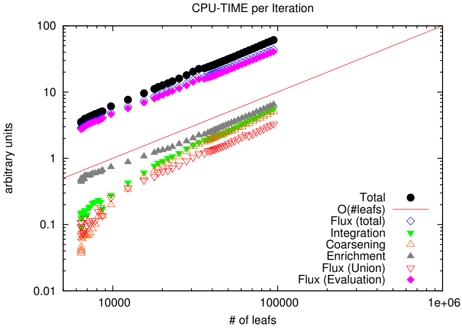
L^2-norm of stochastic error for different values of $\eta \in [10^{-2}, 10^{-5}]$ and polynomial degrees No





Left : error as a function of the total number of leafs in the final discretization ($t^n = 0.5$). Right : error as a function of the total number of degrees of freedom (number of leafs times the dimension of the local polynomial basis).





Computational time (per time-iteration) as a function of the stochastic discretization (total number of leafs) ; left : $\mathrm{No} = 2$ and $\eta = 10^{-3}$; right : $\mathrm{No} = 3$ and $\eta = 10^{-4}$.

-Thank you for your attention-

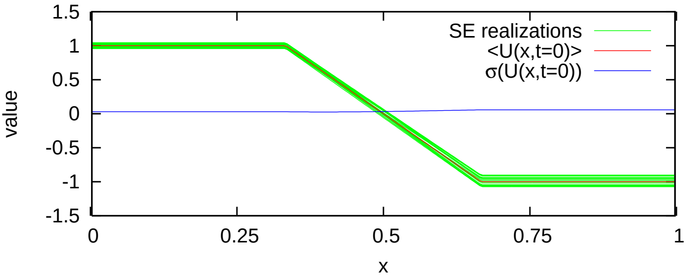
Collaborations :
O. Knio (Duke University), H. Najm and B. Debusschere (Sandia, Livermore) and R. Ghanem (USC)
A. Ern (CEMICS, ENPC), M. Ndjinga and J.-M. Martinez (CEA, Saclay), A. Nouy (GeM, Centrale Nantes), L. Mathelin (LIMSI, CNRS)
Current Fundings :
GNR MoMaS and Agence Nationale pour la Recherche (projects : TYCHE & ASRMEI)

2nd test case

Continuous initial conditions : two constants stochastic states

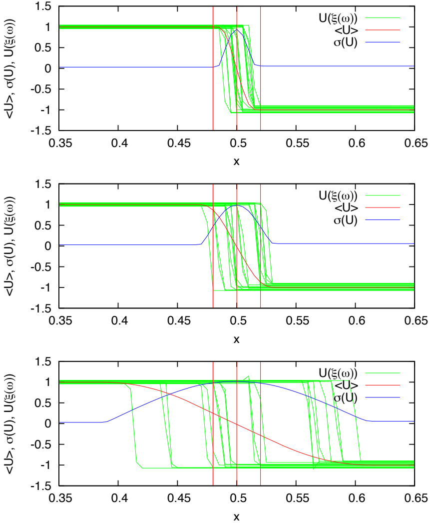
$$U = U^+ = 1 \pm 0.05 \qquad x < 1/3,$$
$$U = U^- = -1 \pm 0.1 \qquad x > 2/3,$$

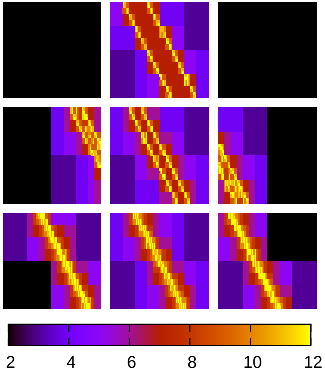
and affine variation in between. $U^+ > U^-$ a.s. and U^+ and U^- independent with uniform distribution : $U^+(\xi_1),\ U^-(\xi_2)$.



$U^+(\xi_1) + U^-(\xi_2) \neq 0$ a.s.

2nd test case





Solution with *x* at different times.

Convergence with max resolution level (8 to 14)