Parametric Problems, Stochastics, and Identification

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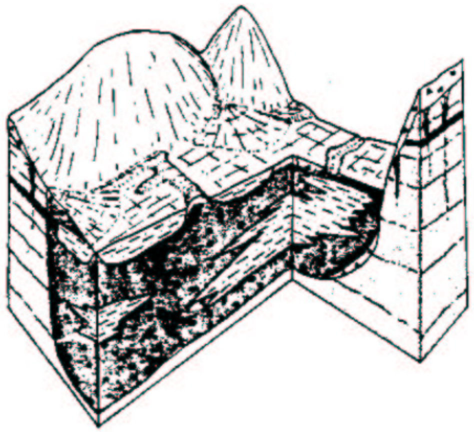
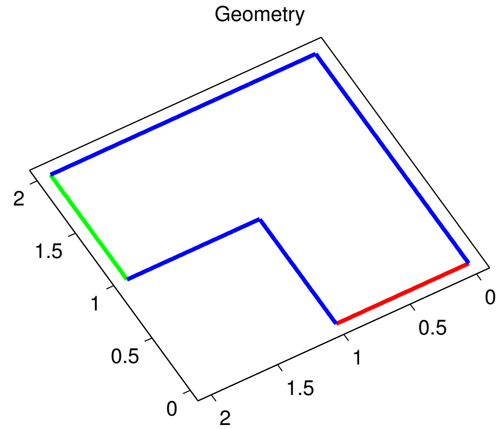
Overview

1. Parameter identification
2. Parametric forward problem
3. Bayesian updating, inverse problems
4. Tensor approximation
5. Bayesian computation
6. Examples

To fix ideas: example problem



Aquifer



2D Model

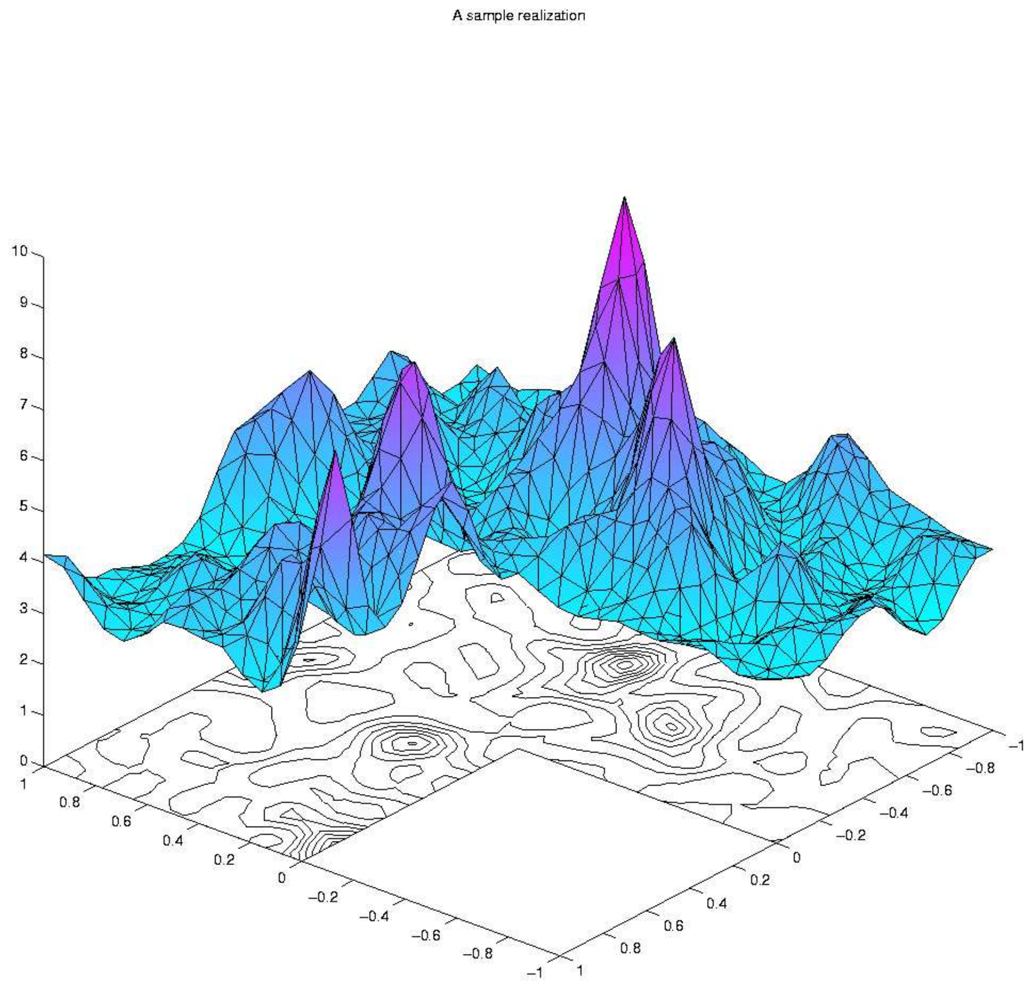
Simple stationary model of groundwater flow with stochastic data

$$-\nabla_x \cdot (\kappa(x,\omega)\nabla_x u(x,\omega)) = f(x,\omega) \quad \text{\& b.c.,} \qquad x \in \mathcal{G} \subset \mathbb{R}^d$$

$$-\kappa(x,\omega)\nabla_x u(x,\omega) = g(x,\omega), \qquad x \in \Gamma \subset \partial\mathcal{G}, \quad \omega \in \Omega.$$

Parameter $q(x,\omega) = \log \kappa(x,\omega)$ is uncertain,
the stochastic conductivity κ, as well as f and g — sinks and sources.

Realisation of $\kappa(x, \omega)$



Mathematical setup

Consider operator equation, physical system modelled by A:

$$A(u) = f \qquad u \in \mathcal{U},\ f \in \mathcal{F},$$

$$\Leftrightarrow \forall v \in \mathcal{U}: \qquad \langle A(u), v\rangle = \langle f, v\rangle,$$

$\mathcal{U}$ — space of states, $\mathcal{F} = \mathcal{U}^*$ — dual space of actions / forcings.

Solution operator: $u = U(f)$, inverse of A.

Operator depends on parameters $q \in \mathcal{Q}$,
hence state u is also function of q:

$$A(u; q) = f(q) \quad \Rightarrow \quad u = U(f; q).$$

Measurement operator Y with values in $\mathcal{Y}$:

$$y = Y(q; u) = Y(q, U(f; q)).$$

Forward parametric problem

Parametric elements: operator $A(\cdot;q)$, rhs $f(q)$, state $u(q)$, $\to r(q)$.

Goal are representations of $r(q) \in \mathcal{W}$, i.e. $r : \mathcal{Q} \to \mathcal{W}$.
Help from inner product $\langle\cdot|\cdot\rangle_{\mathcal{R}}$ on subspace $\mathcal{R} \subset \mathbb{R}^{\mathcal{Q}}$.
In case $\mathcal{Q}$ is a measure / probability space, $\mathcal{R} = L_2$.

To each parametric element corresponds linear map

$$R : \mathcal{W} \ni \hat{r} \mapsto \langle \hat{r} | r(\cdot) \rangle_{\mathcal{R}} \in \mathcal{R}.$$

Key is self-adjoint positive map $C = R^* R : \mathcal{W} \to \mathcal{W}$.
Spectral factorisation of C leads to Karhunen-Loève representation,
a tensor product rep., corresponds to SVD of R (a.k.a. POD).

Each factorisation $C = B^* B$ leads to a tensor representation,
(ex.: smoothed white noise)
a 1–1 correspondence between factorisations and representations.

Setting for the identification process

General idea:

We observe / measure a system, whose structure we know in principle.
The system behaviour depends on some quantities (parameters),
which we do not know $\Rightarrow$ uncertainty.

We model (uncertainty in) our knowledge in a Bayesian setting:
as a probability distribution on the parameters.

We start with what we know a priori, then perform a measurement.
This gives new information, to update our knowledge (identification).

Update in probabilistic setting works with conditional probabilities
$\Rightarrow$ Bayes's theorem.

Repeated measurements lead to better identification.

Inverse problem

For given f, measurement y is just a function of q.
This function is usually not invertible $\Rightarrow$ ill-posed problem,
measurement y does not contain enough information.

In Bayesian framework state of knowledge modelled in a probabilistic way,
parameters q are uncertain, and assumed as random.

Bayesian setting allows updating / sharpening of information
about q when measurement is performed.

The problem of updating distribution—state of knowledge of q
becomes well-posed.

Can be applied successively, each new measurement y and
forcing f —may also be uncertain—will provide new information.

Model with uncertainties

For simplicity assume that $\mathcal{Q}$ is a Hilbert space,
and $q(\omega)$ has finite variance — $\|q\|_{\mathcal{Q}} \in \mathcal{S} := L_2(\Omega)$, so that

$$q \in L_2(\Omega, \mathcal{Q}) \cong \mathcal{Q} \otimes L_2(\Omega) = \mathcal{Q} \otimes \mathcal{S} =: \mathscr{Q}.$$

System model is now

$$A(u(\omega); q(\omega)) = f(\omega) \qquad \text{a.s. in } \omega \in \Omega,$$

state $u = u(\omega)$ becomes $\mathcal{U}$-valued random variable (RV),
element of a tensor space $\mathscr{U} = \mathcal{U} \otimes \mathcal{S}$.

As variational statement:

$$\forall v \in \mathscr{U} : \quad \mathbb{E}\left(\langle A(u(\cdot); q(\cdot)), v\rangle\right) = \mathbb{E}\left(\langle f(\cdot), v\rangle\right) =: \langle\langle f, v\rangle\rangle.$$

Leads to well-posed stochastic PDE (SPDE).

Representation of randomness

Parameters q modelled as $\mathcal{Q}$-valued (a vector space) RVs on some probability space $(\Omega, \mathbb{P}, \mathfrak{A})$, with expectation operator $\mathbb{E}\left(q\right) = \bar{q}$.

RVs $q : \Omega \to \mathcal{Q}$ (and $u(q)$) may be represented in the following ways:

Samples: the best known representation, i.e. $q(\omega_1), \ldots, q(\omega_N), \ldots$

Distribution of q. This is the push-forward measure $q_*\mathbb{P}$ on $\mathcal{Q}$.

Moments of q, like $\mathbb{E}\left(q \otimes \ldots \otimes q\right)$ (mean, covariance, ...).

Functional/Spectral: Functions of other (known) RVs, like Wiener's polynomial chaos, i.e. $q(\omega) = q(\theta_1(\omega)), \ldots, \theta_M(\omega), \ldots) =: q(\boldsymbol{\theta})$.

Sampling and functional representation work with vectors, allows linear algebra in computation.

Computational approaches

Representation determines algorithms:

- **Distributions** $\longrightarrow$ Kolmogorov / Fokker-Planck equations.
  Needs new software, deterministic solver $u = S(f, q)$ not used.
- **Moments** $\longrightarrow$ New (sometimes difficult) equations.
  Needs new software, deterministic solver mostly not used.
- **Sampling** $\longrightarrow$ Domain of direct integration methods;
  (quasi) Monte Carlo, sparse (Smolyak) grids, etc.
  Obviously non-intrusive; software interface $\to$ solve.
- **Functional / Spectral** $\longrightarrow$
  (1) Interpolation / collocation. Based on samples of solution,
  non-intrusive, solve interface.
  (2) Galerkin at first sight intrusive, but with quadrature is also
  non-intrusive, precond. residual interface. Allows greedy rank-1

Conditional probability and expectation

With state $u \in \mathscr{U} = \mathcal{U} \otimes \mathcal{S}$ a RV, the quantity to be measured

$$y(\omega) = Y(q(\omega), u(\omega))) \in \mathscr{Y} := \mathcal{Y} \otimes \mathcal{S}$$

is also uncertain, a random variable.

A new measurement z is performed, composed from the "true" value $y \in \mathcal{Y}$ and a random error ϵ: $z(\omega) = y + \epsilon(\omega) \in \mathscr{Y}$.

Classically, Bayes's theorem gives conditional probability

$$\mathbb{P}(I_q|M_z) = \frac{\mathbb{P}(M_z|I_q)}{\mathbb{P}(M_z)}\mathbb{P}(I_q);$$

expectation with this posterior measure is conditional expectation.

Kolmogorov starts from conditional expectation $\mathbb{E}(\cdot|M_z)$, from this conditional probability via $\mathbb{P}(I_q|M_z) = \mathbb{E}\left(\chi_{I_q}|M_z\right)$.

Update

The conditional expectation is defined as
orthogonal projection onto the closed subspace $L_2(\Omega, \mathbb{P}, \sigma(z))$:

$$\mathbb{E}(q|\sigma(z)) := P_{\mathscr{Q}_\infty} q = \text{argmin}_{\tilde{q} \in L_2(\Omega, \mathbb{P}, \sigma(z))} \|q - \tilde{q}\|^2_{L_2}$$

The subspace $\mathscr{Q}_\infty := L_2(\Omega, \mathbb{P}, \sigma(z))$ represents the available information, estimate minimises $\Phi(\cdot) := \|q - (\cdot)\|^2$ over $\mathscr{Q}_\infty$.
More general loss functions than mean square error are possible.

The update, also called the assimilated value
$q_a(\omega) := P_{\mathscr{Q}_\infty} q = \mathbb{E}(q|\sigma(z))$, is a $\mathcal{Q}$-valued RV
and represents new state of knowledge after the measurement.

Reduction of variance—Pythagoras: $\|q\|^2_{L_2} = \|q - q_a\|^2_{L_2} + \|q_a\|^2_{L_2}$
Doob-Dynkin: $\mathscr{Q}_\infty = \{\varphi \in \mathscr{Q} \; : \; \varphi = \phi \circ Y, \phi \text{ measurable } \}$

Important points I

The probability measure $\mathbb{P}$ is not the object of desire.
It is the distribution of q, a measure on $\mathcal{Q}$—push forward of $\mathbb{P}$:

$$q_*\mathbb{P}(\mathcal{E}) := \mathbb{P}(q^{-1}(\mathcal{E})) \quad \text{for measurable} \quad \mathcal{E} \subseteq \mathcal{Q}.$$

Bayes's original formula changes $\mathbb{P}$, leaves q as is.
Kolmogorov's conditional expectation changes q, leaves $\mathbb{P}$ as is.
In both cases the update is a new $q_*\mathbb{P}$.

$\mathbb{P}$ (a probability measure) is on positive part of unit sphere,
whereas q is free in a vector space.

This will allow the use of (multi-)linear algebra
and tensor approximations.

Important points II

Identification process:

- Use forward problem $A(u(\omega); q(\omega)) = f(\omega)$ to forecast new state $u_f(\omega)$ and measurement $y_f(\omega) = Y(q(\omega), u_f(\omega)))$.
- Perform minimisation of loss function to obtain update map / filter.
- Use innovation in inverse problem to update forecast q_f to obtain assimilated (updated) q_a with update map.
- All operations in vector space, use tensor approximations throughout.

Approximation

Minimisation equivalent to orthogonality: find $\phi \in L_0(\mathcal{Y}, \mathcal{Q})$

$$\forall p \in \mathscr{Q}_\infty : \quad \langle\!\langle \mathrm{D}_{q_a} \Phi(q_a(\phi)), p \rangle\!\rangle_{L_2} = \langle\!\langle q - q_a, p \rangle\!\rangle_{L_2} = 0,$$

Approximation of $\mathscr{Q}_\infty$: take $\mathscr{Q}_n \subset \mathscr{Q}_\infty$

$$\mathscr{Q}_n := \{\varphi \in \mathscr{Q} \ : \ \varphi = \psi_n \circ Y,\ \psi_n \text{ a } n^{\text{th}} \text{ degree polynomial}\}$$

i.e. $\varphi = {}^0H + {}^1HY + \cdots + {}^kHY^{\otimes k} + \cdots + {}^nHY^{\otimes n}$,
where ${}^kH \in \mathscr{L}_s^k(\mathcal{Y}, \mathcal{Q})$ is symmetric and k-linear.

With $q_a(\phi) = q_a(({}^0H, \ldots, {}^kH, \ldots, {}^nH)) = \sum_{k=0}^{n} {}^kH\, z^{\otimes k} = P_{\mathscr{Q}_n} q$,
orthogonality implies

$$\forall \ell = 0, \ldots, n : \quad \mathrm{D}_{({}^\ell H)} \Phi(q_a({}^0H, \ldots, {}^kH, \ldots, {}^nH)) = 0$$

Determining the n-th degree Bayesian update

With the abbreviations

$$\langle p \otimes v^{\otimes k} \rangle := \mathbb{E}\left(p \otimes v^{\otimes k}\right) = \int_{\Omega} p(\omega) \otimes v(\omega)^{\otimes k} \, \mathbb{P}(\mathrm{d}\omega),$$

and ${}^{k}H \langle z^{\otimes(\ell+k)} \rangle := \langle z^{\otimes \ell} \otimes ({}^{k}H \, z^{\otimes k}) \rangle = \mathbb{E}\left(z^{\otimes \ell} \otimes ({}^{k}H \, z^{\otimes k})\right)$,

we have for the unknowns $({}^{0}H, \ldots, {}^{k}H, \ldots, {}^{n}H)$

$$\begin{array}{lllll} \ell = 0: & {}^{0}H & \cdots + {}^{k}H \langle z^{\otimes k} \rangle & \cdots + {}^{n}H \langle z^{\otimes n} \rangle = & \langle q \rangle, \\ \ell = 1: & {}^{0}H \langle z \rangle & \cdots + {}^{k}H \langle z^{\otimes(1+k)} \rangle & \cdots + {}^{n}H \langle z^{\otimes(1+n)} \rangle = & \langle q \otimes z \rangle, \\ \vdots & & \ldots & \vdots & \vdots \\ \ell = n: & {}^{0}H \langle z^{\otimes n} \rangle & \cdots + {}^{k}H \langle z^{\otimes(n+k)} \rangle & \cdots + {}^{n}H \langle z^{\otimes 2n} \rangle = & \langle q \otimes z^{\otimes n} \rangle \end{array}$$

a linear system with symmetric positive definite Hankel operator matrix $(\langle z^{\otimes(\ell+k)} \rangle)_{\ell,k}$.

Bayesian update in components

For short $\forall \ell = 0, \ldots, n:$

$$\sum_{k=0}^{n} {}^{k}H \langle z^{\otimes(\ell+k)} \rangle = \langle q \otimes z^{\otimes \ell} \rangle,$$

For finite dimensional spaces, or after discretisation,
in components (or à la Penrose in 'symbolic index' notation):
let $q = (q^m)$, $z = (z^\jmath)$, and ${}^{k}H = ({}^{k}H^{m}_{\jmath_1 \cdots \jmath_k})$, then:

$\forall \ell = 0, \ldots, n;$

$$\langle z^{\jmath_1} \cdots z^{\jmath_\ell} \rangle ({}^{0}H^{m}) + \cdots + \langle z^{\jmath_1} \cdots z^{\jmath_{\ell+1}} \cdots z^{\jmath_{\ell+k}} \rangle ({}^{k}H^{m}_{\jmath_{\ell+1} \cdots \jmath_{\ell+k}}) +$$
$$\cdots + \langle z^{\jmath_1} \cdots z^{\jmath_{\ell+1}} \cdots z^{\jmath_{\ell+n}} \rangle ({}^{n}H^{m}_{\jmath_{\ell+1} \cdots \jmath_{\ell+n}}) = \langle q^m z^{\jmath_1} \cdots z^{\jmath_\ell} \rangle.$$

(Einstein summation convention used)

matrix does not depend on m—it is identically block diagonal.

Special cases

For $n = 0$ (constant functions) $\Rightarrow\; q_a = {}^0H = \langle q \rangle \quad (= \mathbb{E}(q))$.

For $n = 1$ the approximation is with affine functions

$$\begin{aligned} {}^0H \quad + {}^1H \langle z \rangle \quad &= \langle q \rangle \\ {}^0H \langle z \rangle + {}^1H \langle z \otimes z \rangle &= \langle q \otimes z \rangle \end{aligned}$$

$\Longrightarrow$ (remember that $[\mathrm{cov}_{qz}] = \langle q \otimes z \rangle - \langle q \rangle \otimes \langle z \rangle$)

$$\begin{aligned} {}^0H = \quad & \langle q \rangle - {}^1H \langle z \rangle \\ {}^1H (\langle z \otimes z \rangle - \langle z \rangle \otimes \langle z \rangle) = & \langle q \otimes z \rangle - \langle q \rangle \otimes \langle z \rangle \end{aligned}$$

$\Rightarrow\; {}^1H = [\mathrm{cov}_{qz}][\mathrm{cov}_{zz}]^{-1}$ (Kalman's solution),

${}^0H = \langle q \rangle - [\mathrm{cov}_{qz}][\mathrm{cov}_{zz}]^{-1} \langle z \rangle$,

and finally

$$q_a = {}^0H + {}^1H\, z = \langle q \rangle + [\mathrm{cov}_{qz}][\mathrm{cov}_{zz}]^{-1}(z - \langle z \rangle).$$

Case with prior information

Here we have prior information $\mathscr{Q}_f$ and prior estimate $q_f(\omega)$ (forecast) and measurements z generating via Y a subspace $\mathscr{Q}_y \subset \mathscr{Q}$.

We now need projection onto $\mathscr{Q}_a = \mathscr{Q}_f + \mathscr{Q}_y$, with reformulation as an orthogonal direct sum:

$$\mathscr{Q}_a = \mathscr{Q}_f + \mathscr{Q}_y = \mathscr{Q}_f \oplus (\mathscr{Q}_y \cap \mathscr{Q}_f^\perp) = \mathscr{Q}_f \oplus \mathscr{Q}_\infty.$$

The update / conditional expectation / assimilated value is the orthogonal projection

$$q_a = q_f + P_{\mathscr{Q}_\infty} q = q_f + q_\infty,$$

where q_∞ is the innovation.

Compute q_a by approximating: $\mathscr{Q}_n \subset \mathscr{Q}_\infty$. We now take $n = 1$.

Simplification

The case $n = 1$—linear functions, projecting onto $\mathscr{Q}_1$—is well known:

this is the linear minimum variance estimate $\hat{q}_a$.

Theorem: (Generalisation of Gauss-Markov)

$$\hat{q}_a(\omega) = q_f(\omega) + {}^1H(z(\omega) - y_f(\omega)),$$

where the linear Kalman gain operator ${}^1H : \mathscr{Y} \to \mathscr{Q}$ is

$${}^1H := [\mathrm{cov}_{qz}][\mathrm{cov}_{zz}]^{-1} = [\mathrm{cov}_{qy}][\mathrm{cov}_{yy} + \mathrm{cov}_{\epsilon\epsilon}]^{-1}.$$

(The normal Kalman filter is a special case.)

Or in tensor space $q \in \mathscr{Q} = \mathcal{Q} \otimes \mathcal{S}$:

$$\hat{q}_a = q_f + ({}^1H \otimes I)(z - y_f).$$

Deterministic model, discretisation, solution

Remember operator equation: $A(u) = f \qquad u \in \mathcal{U},\ f \in \mathcal{F}.$

Solution is usually by first discretisation

$$\boldsymbol{A}(\boldsymbol{u}) = \boldsymbol{f} \qquad \boldsymbol{u} \in \mathcal{U}_N \subset \mathcal{U},\ \boldsymbol{f} \in \mathcal{F}_N = \mathcal{U}_N^* \subset \mathcal{F},$$

and then (iterative) numerical solution process

$$\boldsymbol{u}_{k+1} = \boldsymbol{S}(\boldsymbol{u}_k), \qquad \lim_{k\to\infty} \boldsymbol{u}_k = \boldsymbol{u}.$$

$\boldsymbol{S}$ evaluates (pre-conditioned) residua $\boldsymbol{f} - \boldsymbol{A}(\boldsymbol{u}_k)$.

Similarly for model with uncertainty:

$$\boldsymbol{A}(\boldsymbol{u}(\omega); \boldsymbol{q}(\omega)) = \boldsymbol{f}(\omega),$$

assume $\{\boldsymbol{v}_j\}_{j=1}^N$ a basis in $\mathcal{U}_N$, then the approx. solution in $\mathcal{U}_N \otimes \mathcal{S}$

$$\boldsymbol{u}(\omega) = \sum_{j=1}^{N} u_j(\omega)\boldsymbol{v}_j.$$

Discretisation by functional approximation

Choose subspace $\mathcal{S}_B \subset \mathcal{S}$ with basis $\{X_\beta\}_{\beta=1}^{B}$,
make ansatz for each $u_j(\omega) \approx \sum_\beta u_j^\beta X_\beta(\omega)$, giving

$$\boldsymbol{u}(\omega) = \sum_{j,\beta} u_j^\beta X_\beta(\omega) \boldsymbol{v}_j = \sum_{j,\beta} u_j^\beta X_\beta(\omega) \otimes \boldsymbol{v}_j.$$

Solution is in tensor product $\mathscr{U}_{N,B} := \mathcal{U}_N \otimes \mathcal{S}_B \subset \mathcal{U} \otimes \mathcal{S} = \mathscr{U}$.
State $\boldsymbol{u}(\omega)$ represented by tensor $\mathbf{u} := \mathbf{u}_N^B := \{u_j^\beta\}_{j=1,\dots,N}^{\beta=1,\dots,B}$,
(β is usually multi-index)
similarly for all other quantities, fully discrete forward model
is obtained by weighting residual with Ξ_α with ansatz inserted:

$$\forall \alpha : \left\langle \Xi_\alpha(\omega), \boldsymbol{f}(\omega) - \boldsymbol{A}\left(\sum_{j,\beta} u_j^\beta X_\beta(\omega) \boldsymbol{v}_j ; \boldsymbol{q}(\omega)\right)\right\rangle_{\mathcal{S}} = 0.$$

Stochastic forward problem

$\Rightarrow$ generally coupled system of equations for $\mathbf{u} = \{u_j^\beta\}$:

$$\mathbf{A}(\mathbf{u};\mathbf{q}) = \mathbf{f}, \qquad \mathbf{y} = \mathbf{Y}(\mathbf{q};\mathbf{u}).$$

- If $\Xi_\alpha(\cdot) = \delta(\cdot - \omega_\alpha)$, system decouples $\longrightarrow$ collocation / interpolation; may use for each ω_α original solver $\boldsymbol{S}$ (obviously non-intrusive).
- If $\Xi_\alpha(\cdot) = X_\alpha(\cdot) \longrightarrow$ Bubnov-Galerkin conditions; with numerical integration uses also original solver $\boldsymbol{S}$ and is also non-intrusive.
- In greedy rank-one update tensor solver one uses Bubnov-Galerkin conditions (proper gener. decomp. (PGD)/ succ. rank-1 upd. (SR1U)/ alt. least squ. (ALS)), also possible by non-intrusive use of original $\boldsymbol{S}$.

For update: ${}^1\mathbf{H} = {}^1\boldsymbol{H} \otimes \boldsymbol{I}$ computed analytically (X_β =Hermite basis)

$$[\mathrm{cov}_{qy}] = \textstyle\sum_{\alpha>0} \alpha!\, \boldsymbol{q}^\alpha(\boldsymbol{y}^\alpha)^T; \qquad [\mathrm{cov}_{yy}] = \textstyle\sum_{\alpha>0} \alpha!\, \boldsymbol{y}^\alpha(\boldsymbol{y}^\alpha)^T.$$

Important points III

Update formulation in vector spaces.

This makes tensor representation possible .

Parametric problems lead to tensor (or separated) representations.

Sparse approximation by low-rank representation.

Possible for forward problem (progressive or iterative).

Possible for inverse problem.

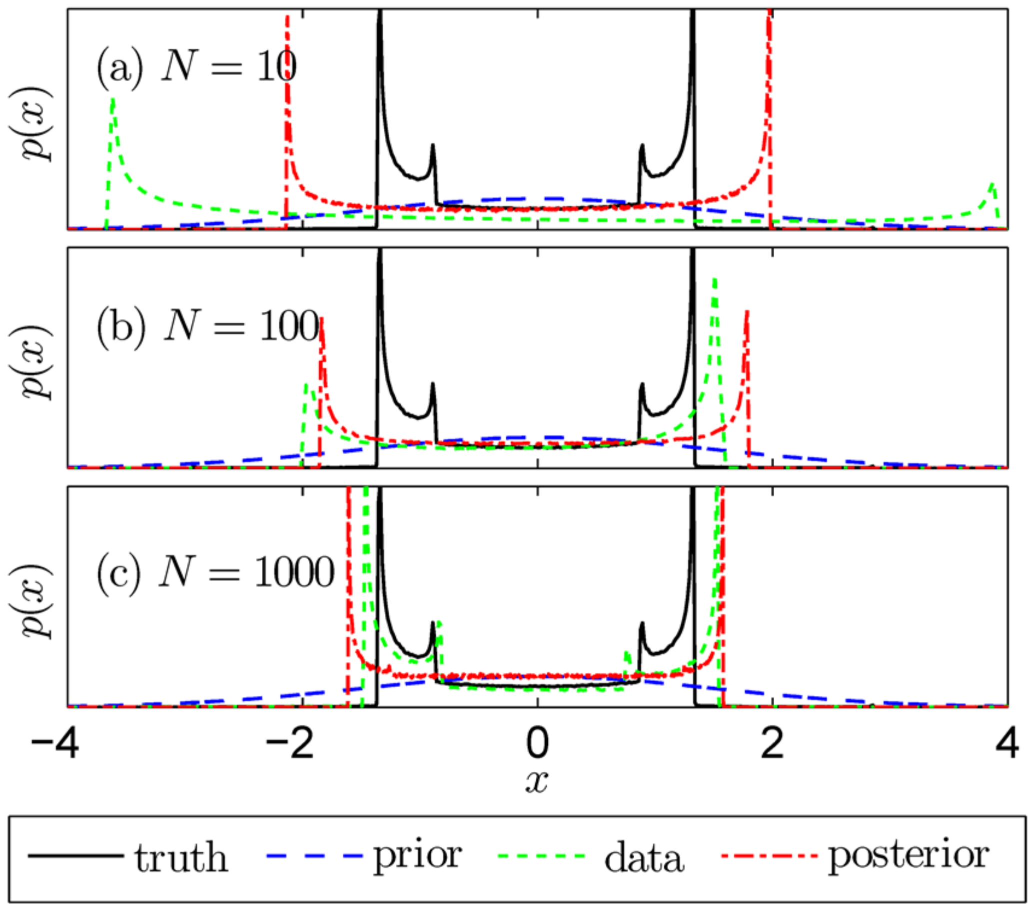
Low-rank approximation can be kept throughout update.

Example 1: Identification of multi-modal dist

Setup: Scalar RV x with non-Gaussian multi-modal "truth" $p(x)$; Gaussian prior; Gaussian measurement errors.

Aim: Identification of $p(x)$.

10 updates of $N = 10, 100, 1000$ measurements.



Example 2: Lorenz-84 chaotic model

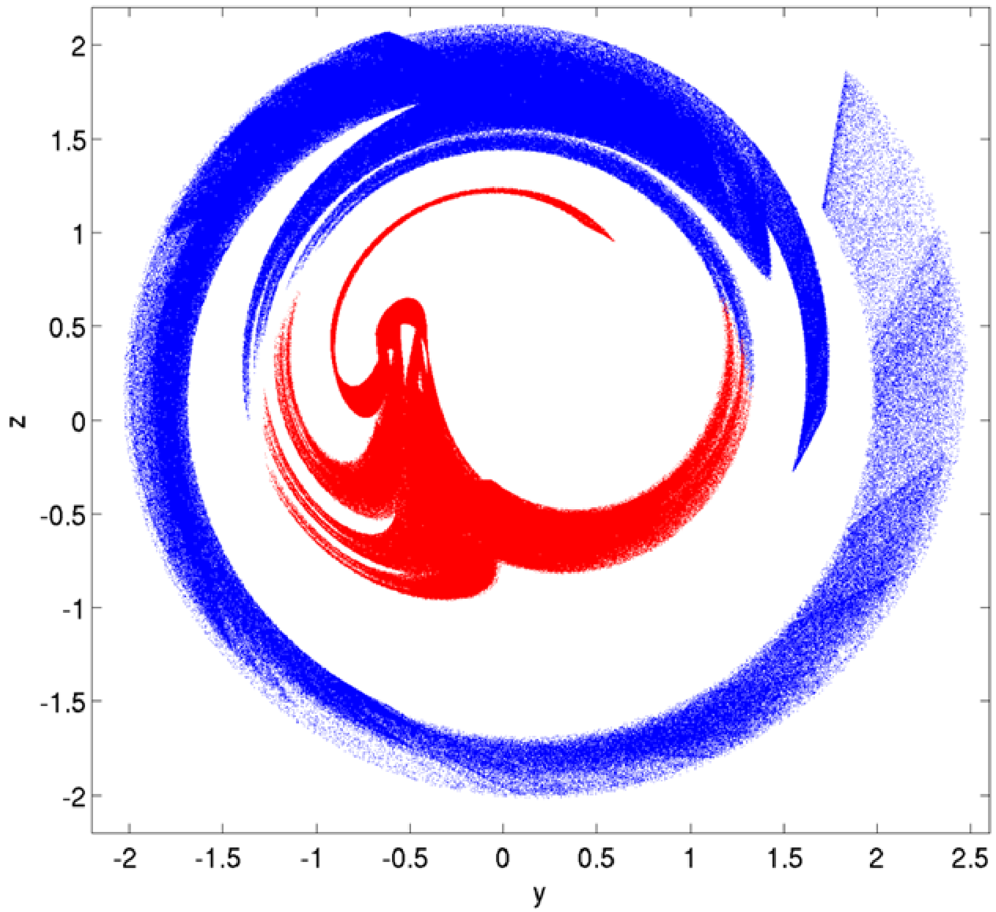
Setup: Non-linear, chaotic system

$$\dot{u} = f(u),\ u = [x, y, z]$$

Small uncertainties in initial conditions u_0 have large impact.

Aim: Sequentially identify state u_t.

Methods: PCE representation and PCE updating and sampling representation and (Ensemble Kalman Filter) EnKF updating.

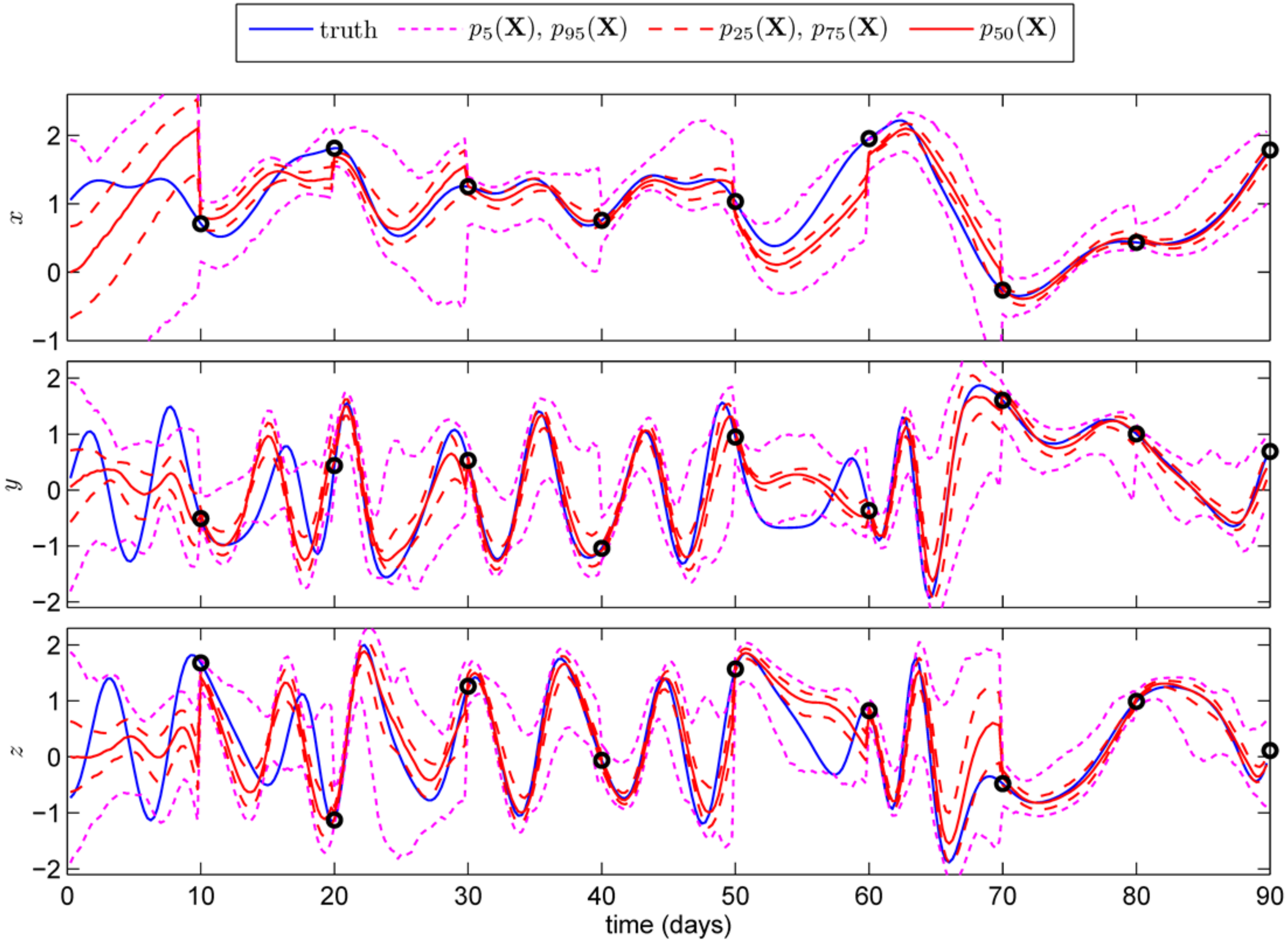


Poincaré cut for $x = 1$.

Example 2: Lorenz-84 PCE representation

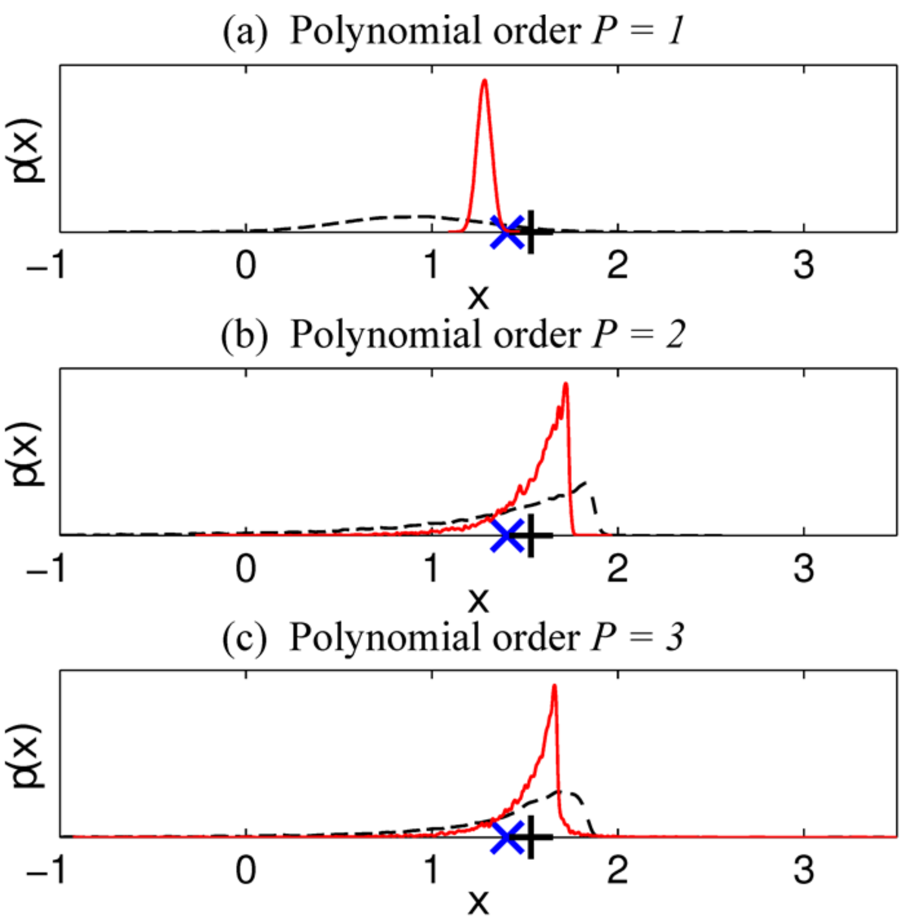
PCE: Variance reduction and shift of mean at update points.

Skewed structure clearly visible, preserved by updates.

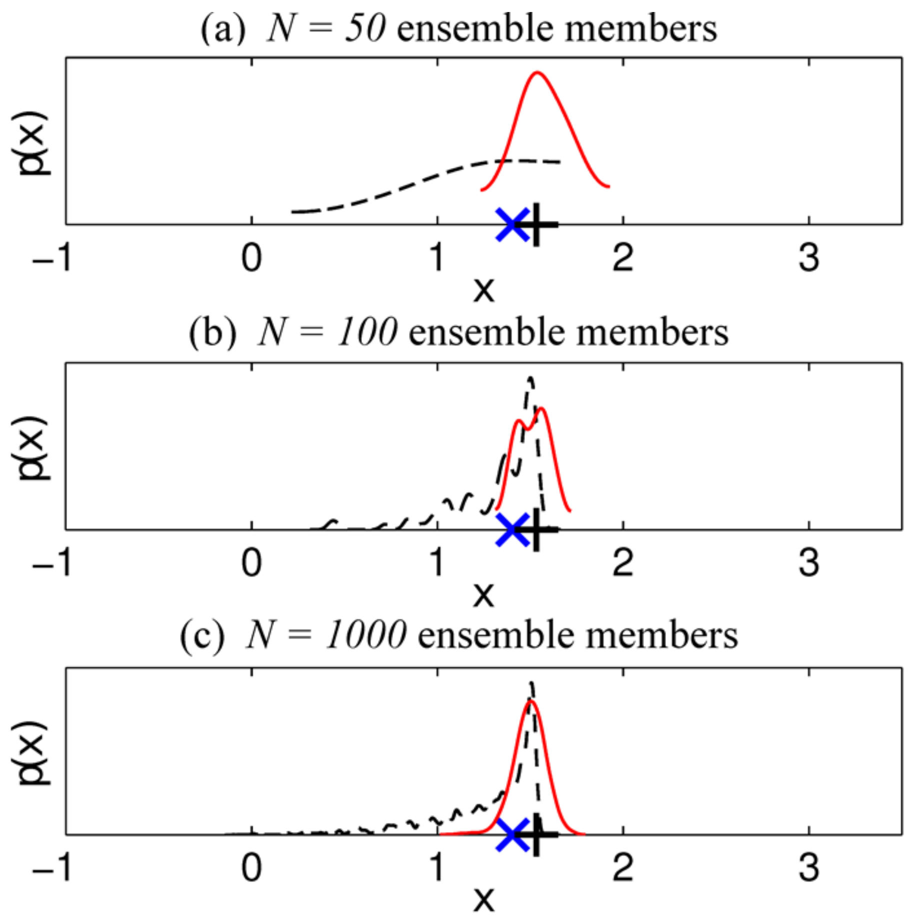


Example 2: Lorenz-84 non-Gaussian identification

PCE

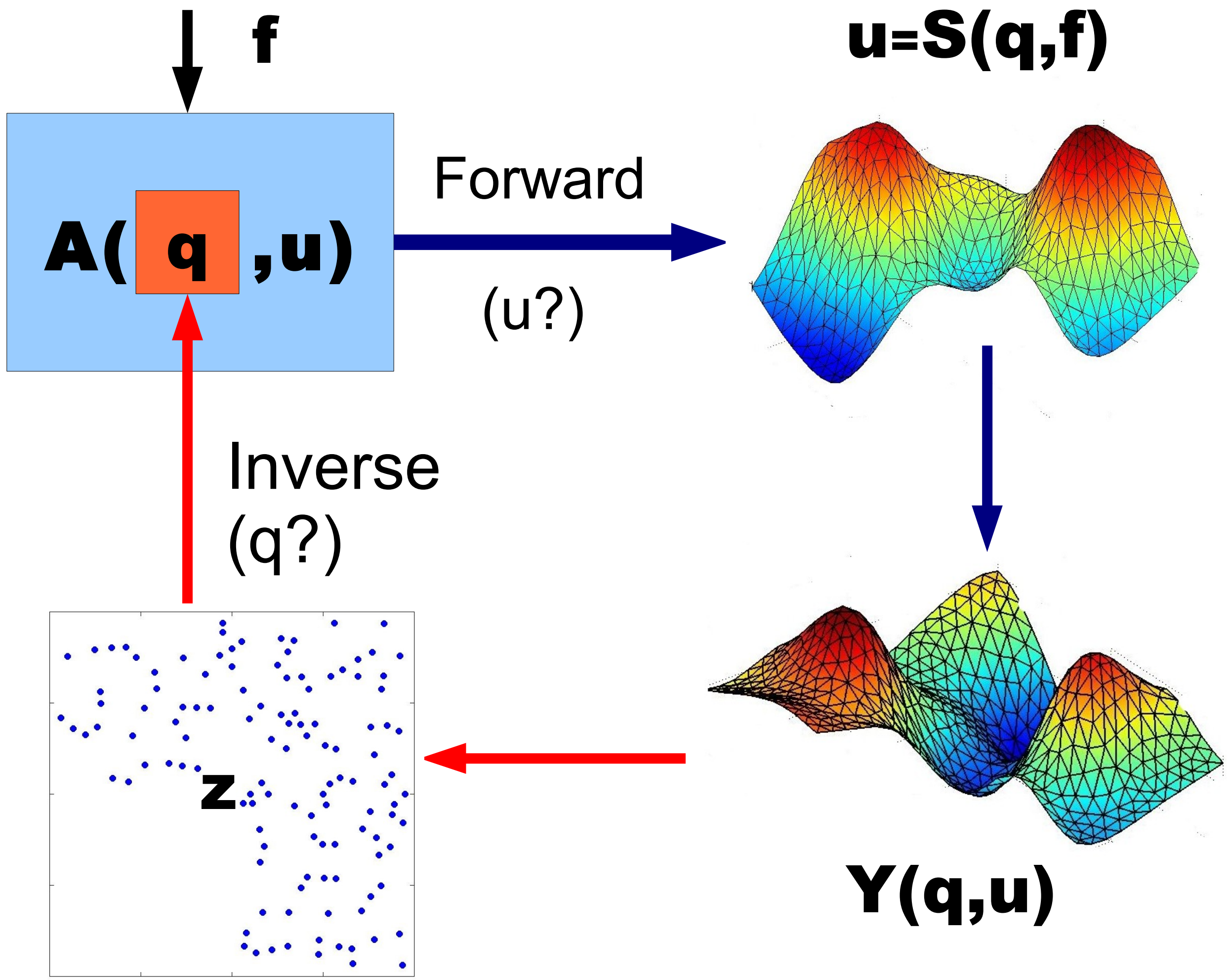


EnKF

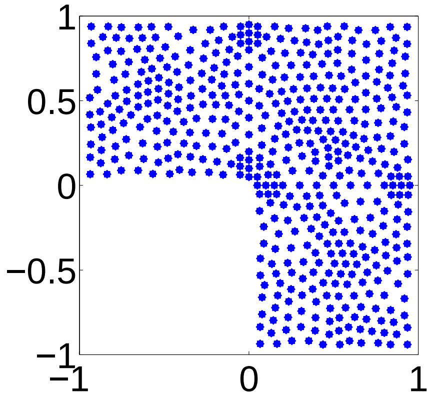


truth $\times$ **measurement** $+$ posterior *prior*

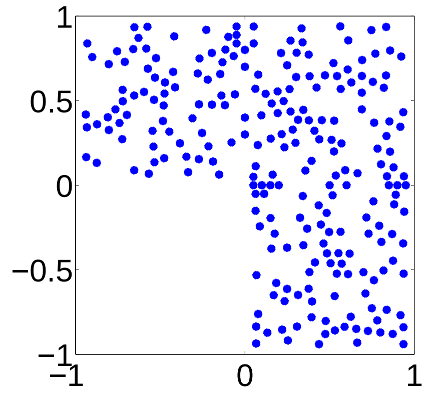
Example 3: diffusion—schematic representation



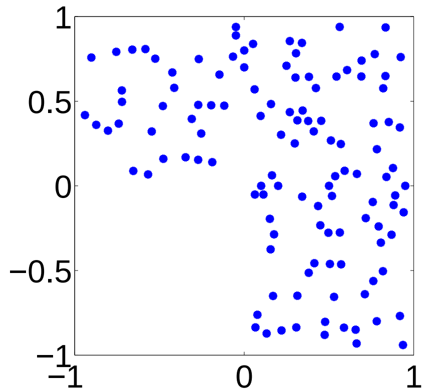
Measurement patches



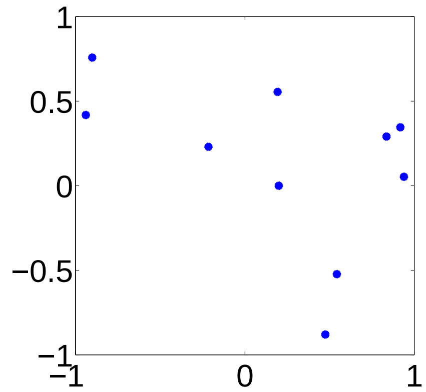
447 measurement patches



239 measurement patches

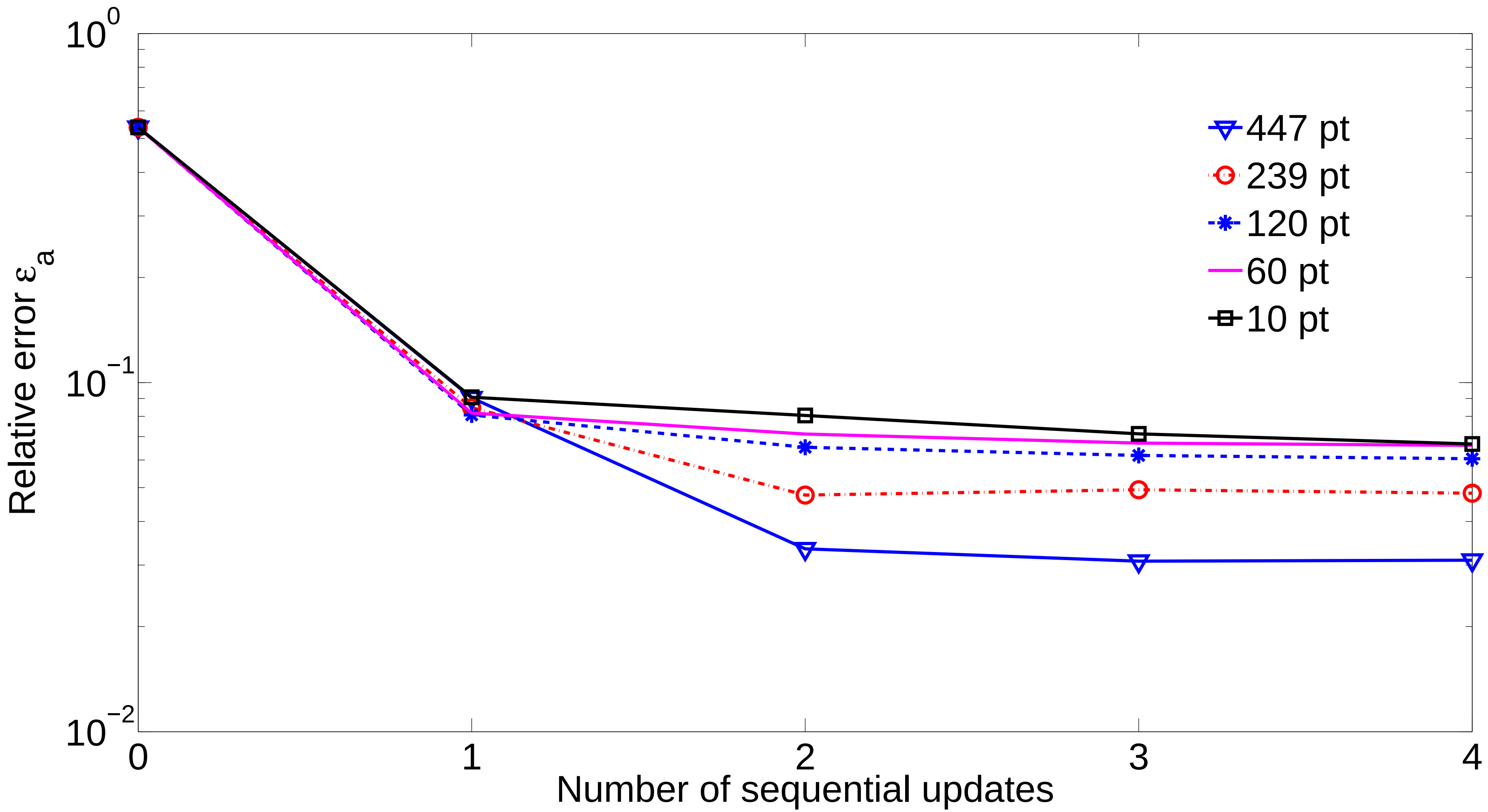


120 measurement patches

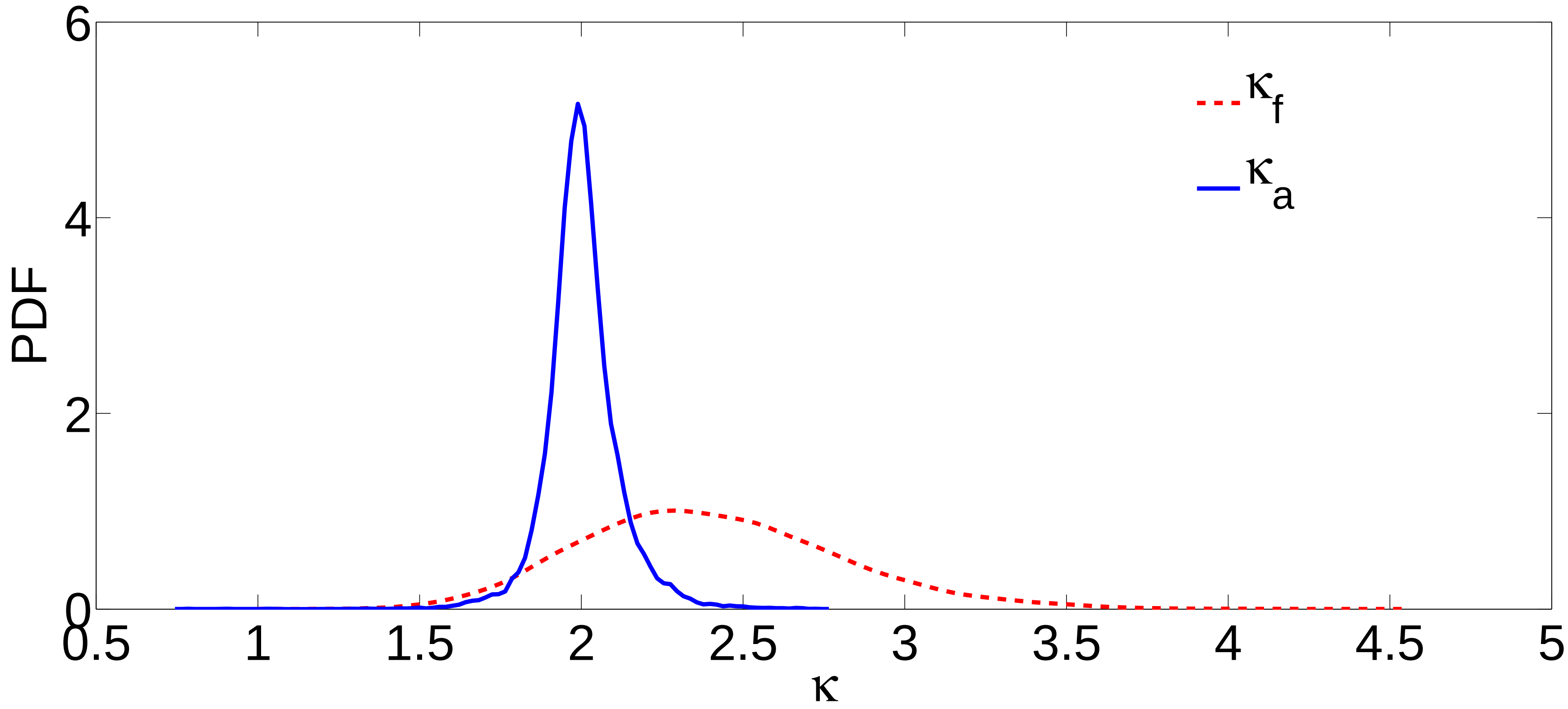


10 measurement patches

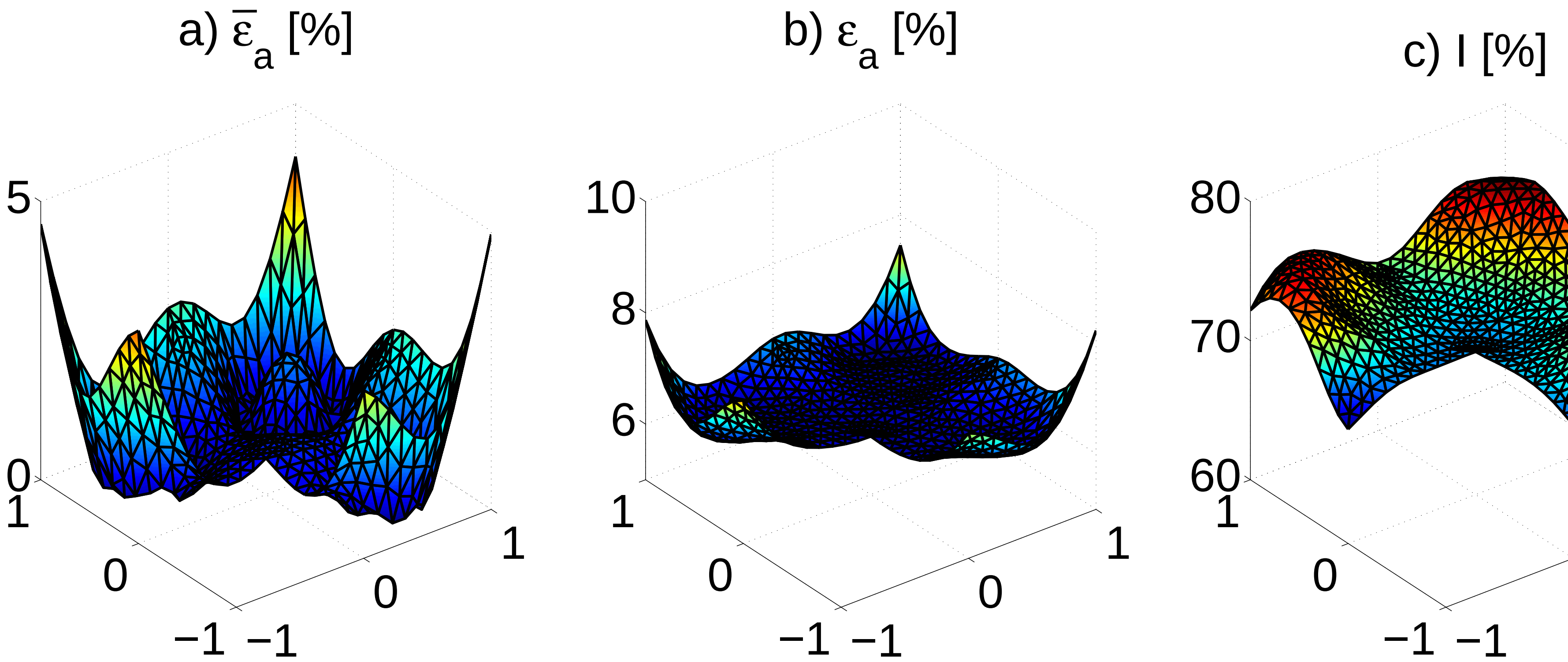
Convergence plot of updates



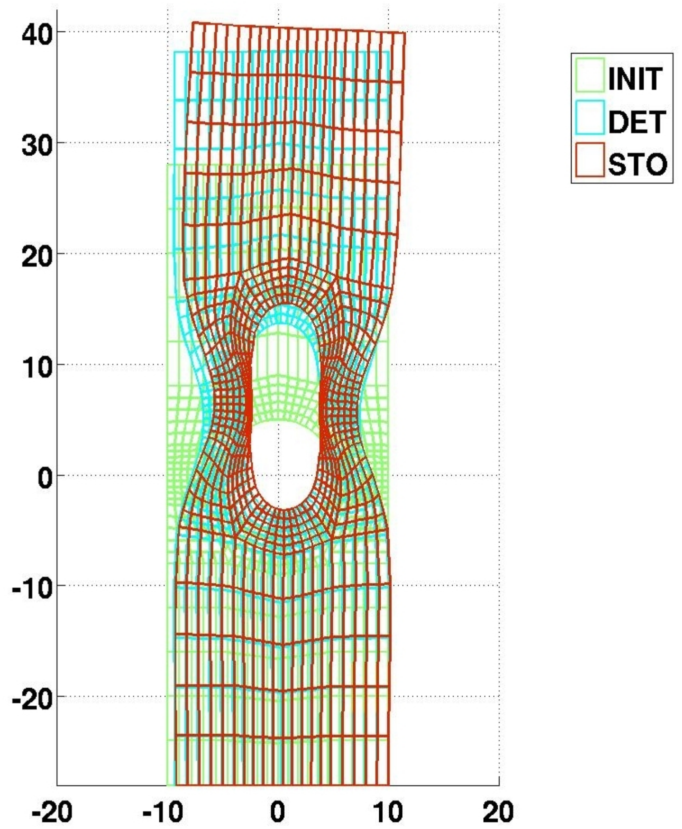
Forecast and Assimilated pdfs



Spatial Error Distribution

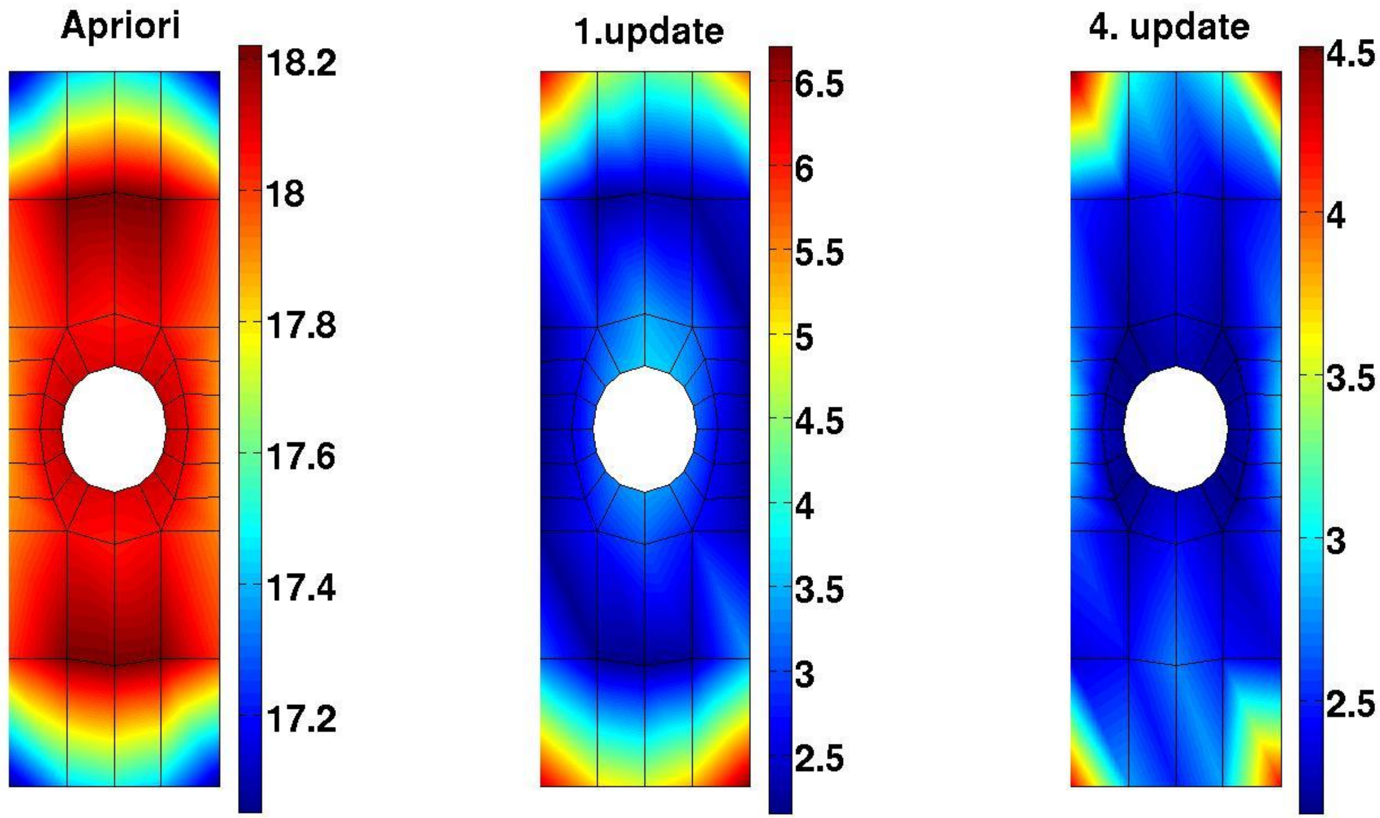


Example 4: plate with hole



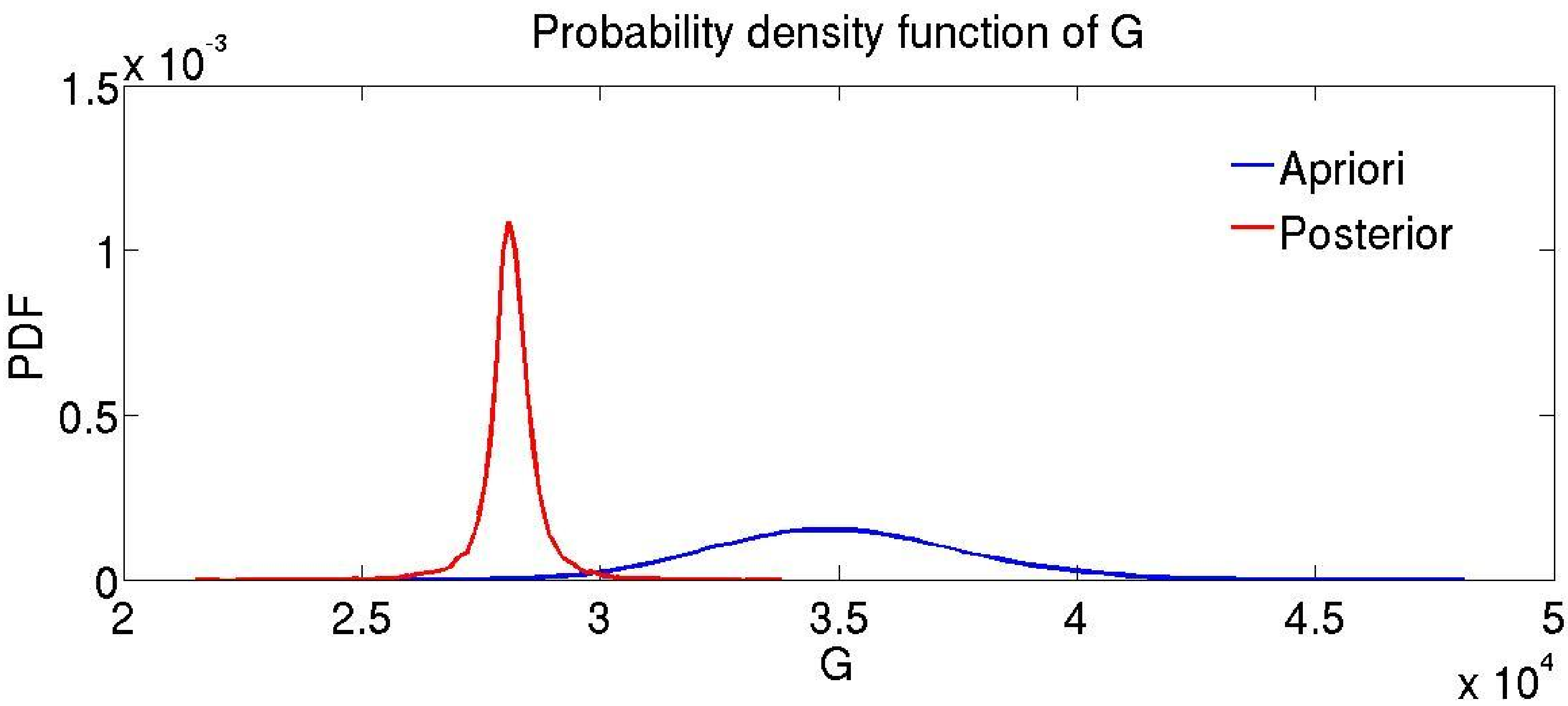
Forward problem: the comparison of the mean values of the total displacement fo r deterministic, initial and stochastic configuration

Relative variance of shear modulus estimate



Relative RMSE of variance [%] after 4th update in 10% equally distributed m easurment points

Probability density shear modulus



Comparison of prior and posterior distribution

Conclusion

- Parametric problems lead to tensor representation.
- Inverse problems via Bayes's theorem.
- Bayesian update is a projection.
- For efficiency try and use sparse representation throughout; ansatz in low-rank tensor products, saves storage as well as computation.
- Bayesian update compatible with low-rank representation.