Approximation of Stochastic PDEs Involving White Noises

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AMSC seminar

Elements of white noise theory

Four examples The pressure equation Linear SPDE with additive and multiplicative noises Helmholtz equation with stochastic refractive index Stochastic schallow water equations

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Conclusions

Fluctuations

- Many physical and engineering models involve
 - uncertain data: forces, sources, initial and boundary conditions, ...

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Fluctuations

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Fluctuations

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In more complex physical models, the data and coefficients are difficult to measure at all locations, and are instead modeled as random fields.

SPDE's involving white noise

Much of the literature on SPDE's, allows for processes with zero correlation length, known as white noise.

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SPDE's involving white noise

- Much of the literature on SPDE's, allows for processes with zero correlation length, known as white noise.
- Pde's perturbed by spatial noise provide an important stochastic model in applications:
 - Pressure equation for fluid in porous media
 - Navier-Stokes equations driven by multiplicative and additive noises
 - Non linear Schrodinger equation with a stochastic potential

SPDE's involving white noise

There are at least two different answers to the questions of how to pose and solve SPDE's involving white noises.

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SPDE's involving white noise

There are at least two different answers to the questions of how to pose and solve SPDE's involving white noises.

The first answer is to blame the roughness of the white noise for the non-solvability of theses equations. Indeed, if the white noise is replaced by colored noise, then there exist ordinary solutions to many SPDE's. But the white noise is a canonical object, this is not the case for colored noise.

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SPDE's involving white noise

The second answer to the question is to use the notion of a generalized solution to SPDEs containing white noise.

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SPDE's involving white noise

- The second answer to the question is to use the notion of a generalized solution to SPDEs containing white noise.
- Walsh has considered a linear SPDE with additive white noise. He showed that for spatial dimension > 1, it is in general not possible to represent the solution as an ordinary stochastic field, but as a distribution (generalized stochastic process).

SPDE's involving white noise

Example : Heat equation driven by a multiplicative space-time white noise W(t, x).

$$u_t = \Delta u + u \dot{W}$$

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This equation has neither weak nor strong solutions in the traditional sense.

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This equation has neither weak nor strong solutions in the traditional sense.

The solution must be defined as a generalized random element

$$x \longrightarrow u(x, \cdot) \in (\mathcal{S})_{-1}, \ x \in \mathbb{R}^d$$

where $(S)_{-1}$ is the Kondratiev space of distribution-valued stochastic processes.

SPDE's involving white noise

This approach has several advantages:



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SPDE's involving white noise

This approach has several advantages:

- SPDEs can be interpreted in the usual strong sense with respect to x.
- ► The space (S)₋₁ is equipped with a multiplication, the Wick product ◊. This gives a natural interpretation of SPDEs where the noise or other terms appear multiplicatively.

Other methods

 Recently, a systematic approach for formulating and discretizing SPDE's with smoothed random data known as SFEM has become popular in the engineering community.

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Other methods

- Recently, a systematic approach for formulating and discretizing SPDE's with smoothed random data known as SFEM has become popular in the engineering community.
- Spectral finite element methods using formal Hermite polynomial chaos (Ghanem, Knio, Le Maitre, Najm, Xiu, Karniadakis,..)
- Collocation finite element methods using tensor product of the space of random variables (Babuška, Tempone, Zouraris, Schwab, ...)

Probability space

The white noise space

$$\Omega = \left(\mathcal{S}'(\mathbb{R}_+ imes \mathbb{R}^d), \mathcal{B}(\mathcal{S}'(\mathbb{R}_+ imes \mathbb{R}^d), \mu
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- \mathcal{B} Borel σ algebra generated by the weak topology in \mathcal{S}' .
- μ: the unique white noise probability measure on B , given by the Bochner-Milnos theorem such that for all f ∈ S(ℝ₊ × ℝ^d)

$$E_{\mu}[e^{i\langle\cdot,f\rangle}] := \int_{\mathcal{S}'} e^{i\langle\omega,f\rangle} d\mu(\omega) = e^{-\frac{1}{2}||f||^2_{L^2(\mathbb{R}_+\times\mathbb{R}^d)}}$$

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Probability space

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$$E_{\mu}[e^{i\langle \cdot, f \rangle}] := \int_{\mathcal{S}'} e^{i\langle \omega, f \rangle} d\mu(\omega) = e^{-\frac{1}{2} \|f\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^d)}^2}$$

• $L^2(\mu) := L^2(\mathcal{S}'(\mathbb{R}_+ imes \mathbb{R}^d), \mathcal{B}, \mu)$ with the inner product

$$(F,G) = E_{\mu}(FG)$$

Fourier-Hermite polynomials

- Let $\{\xi_i \otimes \eta_j\}$ be an orthonormal basis for $L^2(\mathbb{R}_+ \times \mathbb{R}^d)$.
- {η_j}_{j∈ℕ} ⊂ S(ℝ^d) denote the orthonormal basis of L²(ℝ^d) constructed by taking tensor products of Hermite functions.
- {ξ_i}_{i∈ℕ} be the orthonormal basis of L²(ℝ₊) consisting of the Laguerre functions of order ¹/₂.
- Let *I* denote the set of all multi-indices α = (α_{ij}) with α_{ij} ∈ N₀ (i, j ∈ N) with finite length l(α) = max{ij; α_{ij} ≠ 0}.
 For each α ∈ *I* we define the stochastic variable

$$H_lpha(\omega):=\prod_{i,j=1}^{l(lpha)}h_{lpha_{ij}}(\langle\omega,\xi_i\otimes\eta_j
angle).$$

where $h_{\alpha_{ii}}$ are the Hermite polynomials.

Wiener-Itô chaos expansion

The family $\{H_{\alpha} : \alpha \in \mathcal{I}\}$ constitutes an orthogonal basis for $L^{2}(\mu) := L^{2}(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu)$. Then each $f \in L^{2}(\mu)$ has a unique chaos expansion representation:

$$f(t,x,\omega) = \sum_{lpha \in \mathcal{I}} f_{lpha}(t,x) \mathcal{H}_{lpha}(\omega),$$

$$f(t,x,\omega) = f_0(x) + \sum_{lpha \in \mathcal{I}, lpha
eq 0} f_lpha(t,x) H_lpha(\omega)$$

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 $\begin{array}{l} f_{\alpha} = \alpha \text{-th chaos coefficient of } f \\ f_{0} = E_{\mu}[f], \ E_{\mu}[f^{2}] = \sum_{\alpha} \alpha ! \ | \ f_{\alpha} \ |^{2} \end{array}$

Gaussian processes with dependent increments

$$\blacktriangleright m(.,.): \left(\mathbb{R}_+ \times \mathbb{R}^d\right)^2 \longrightarrow \mathbb{R}$$

$$m(.,x) \in L^2(\mathbb{R}_+ \times \mathbb{R}^d) \ , \ rac{\partial^{1+d} m(.,x)}{\partial x_0 \partial x_1 \cdots \partial x_d} \in \mathcal{S}'(\mathbb{R}_+ \times \mathbb{R}^d)$$

and we let:

$$v(y,x) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} m(u,y)m(u,x)du$$

► The stochastic variable with dependent increments B^v(x, ·) is defined by

$$B^{\nu}(x,\omega) := \langle \omega, m(.,x) \rangle = \int_{\mathbb{R}_+ \times \mathbb{R}^d} m(u,x) dB(u,\omega), \ \omega \in \mathcal{S}'(\mathbb{R}_+ \times \mathbb{R}^d)$$

► This process defines a Gaussian process on ℝ₊ × ℝ^d. Its covariance function is given by

$$v(y,x) = \int_{\mathcal{S}'(\mathbb{R}_+ imes \mathbb{R}^d)} B^{v}(x,\omega) B^{v}(y,\omega) d\mu(\omega)$$

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Gaussian processes with dependent increments: examples

Example 1: (Multi-parameter ordinary Brownian motion). If m(u,x) = 1_{[0,x0]×[0,x1]×···×[0,xd]}(u), then the stochastic process B^v(x,ω) is the multi-parameter ordinary Brownian motion B(x,ω) and we have

$$v(y,x) = \prod_{i=0}^{d} \min(x_i, y_i)$$

Example 2: (Multi-parameter fractional Brownian motion)

▶ Let
$$H = (H_0, H_1, \dots, H_d) \in (]0, 1[)^{1+d}$$
 (Hurst vector),
 $f = f_0 \otimes f_1 \otimes \dots \otimes f_d \in S(\mathbb{R}_+ \times \mathbb{R}^d).$
Define

$$M_H f(x) = \prod_{j=0}^d (M_{H_j} f_j)(x_j)$$

where

$$M_{H_j}f_j(x_j) = \begin{cases} K_j \int_{\mathbb{R}} \frac{f_j(x_j - \lambda) - f_j(x_j)}{|\lambda|^{\frac{3}{2} - H_j}} d\lambda & \text{if } 0 < H_j < \frac{1}{2} \\ f_j(x_j) & \text{if } H_j = \frac{1}{2} \\ K_j \int_{\mathbb{R}} \frac{f_j(\lambda)}{|x_i - \lambda|^{\frac{3}{2} - H_j}} d\lambda & \text{if } \frac{1}{2} < H_j < 1 \end{cases}$$

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Example 2: (Multi-parameter fractional Brownian motion)

► The fractional Brownian motion B_H is defined by B_H(x, ω) := ⟨ω, M_H(1_{]0,x0}[...]0,xd[</sub>)⟩ it holds

$$v(y,x) = (rac{1}{2})^{1+d} \prod_{j=0}^{d} \left(\mid x_j \mid^{2H_j} + \mid y_j \mid^{2H_j} - \mid x_j - y_j \mid^{2H_j}
ight)$$

► Example 3: (Gaussian process with short range dependency).
 Let m(u, t) := t² exp (-(u - t)²).
 Hence v(s, t) = √πt²s² exp(-(t - s)²/2) and the process B^v_t is a short range Brownian motion.

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Wiener-Itô chaos expansion

$$B^{\nu}(x,\cdot)\in L^2(\mathcal{S}')$$

• The chaos expansion of Brownian motion B^{ν} is

$$B^{\nu}(x,\omega) = \sum_{i,j=1}^{\infty} (m(\cdot,x),\xi_i \otimes \eta_j)_{L^2(\mathbb{R}_+ \times \mathbb{R}^d)} H_{\epsilon_{ij}}(\omega)$$

Examples:

$$B(x,\omega) = \sum_{i,j=1}^{\infty} \left(\int_0^{x_0} \xi_i(s) ds \int_{-\infty}^{x_d} \cdots \int_{-\infty}^{x_1} \eta_j(y_1,\cdots,y_d) dy_1 \cdots dy_d \right) H_{\epsilon_{ij}}(\omega)$$

$$B_H(x,\omega) = \sum_{i,j=1}^{\infty} \left(\int_0^{x_0} M_H \xi_i(s) ds \int_{-\infty}^{x_d} \cdots \int_{-\infty}^{x_1} M_H \eta_j(y_1,\cdots,y_d) dy_1 \cdots dy_d \right) H_{\epsilon_{ij}}(\omega)$$

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Space of the Kondratiev test functions

 Certain SPDE's involving multiplicative noises don't possess solutions with finite variance.

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Space of the Kondratiev test functions

- Certain SPDE's involving multiplicative noises don't possess solutions with finite variance.
- We must extend our notion of a solution to include solutions with infinite variance in larger space of random elements.

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Space of the Kondratiev test functions

- Certain SPDE's involving multiplicative noises don't possess solutions with finite variance.
- We must extend our notion of a solution to include solutions with infinite variance in larger space of random elements.
- These spaces are the so-called weighted stochastic spaces which include the Hida and Kondratieve spaces and whose elements are characterized by their Wiener chaos coefficients.

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Space of the Kondratiev test functions

► For
$$k = 1, 2, \cdots$$
 and $-1 \le \rho \le 1$, let
 $(S)_{\rho,k} = \left\{ f \in L^2(\mu) : f(\omega) = \sum_{\alpha} c_{\alpha} H_{\alpha}(\omega), c_{\alpha} \in \mathbb{R} \right\}$

such that

$$\|f\|^2_{(\mathcal{S})_{
ho,k}} := \sum_lpha (lpha!)^{1+
ho} c^2_lpha (2\mathbb{N})^{klpha} < \infty$$

where

$$(2\mathbb{N})^{k\alpha} = \prod_{i,j=1}^m (2(i-1)m+j)^{\alpha_{ij}}, \text{ if } \alpha = (\alpha_{ij})_{1 \leq i,j \leq m}$$

The space of Kondratiev test functions (S)_ρ, is defined by

$$(\mathcal{S})_{
ho} = igcap_{k=1}^{\infty} (\mathcal{S})_{
ho,k}$$
Space of Hida distributions

• The space of Hida distributions, $(S)_{-\rho}$, is defined by

$$(S)_{-
ho} = igcup_{k=1}^{\infty} (S)_{-
ho,k}$$

We have

 $(\mathcal{S})_
ho \subset L^2(\mu) \subset (\mathcal{S})_{ho}$

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White noise with dependent increments

$$W^{\nu}(x,\omega) = \frac{\partial^{1+d}}{\partial x_0 \cdots \partial x_d} B^{\nu}(x,\omega) \text{ in } (\mathcal{S})_{-\rho} \text{ for all } x \in \mathbb{R}^{d+1}$$

• We have

$$W^{\nu}(x,.) \in (\mathcal{S})_{-\rho}$$

 $W^{\nu}(x,.) \notin L^{2}(\mu)$

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White noise with dependent increments

Examples:

$$W(x,\omega) = \sum_{i,j} e_i(x_0) \eta_j(x_1,\cdots,x_d) H_{\epsilon_{ij}}(\omega)$$

$$W_H(x,\omega) = \sum_{i,j} M_H e_i(x_0) M_H \eta_j(x_1,\cdots,x_d) H_{\epsilon_{ij}}(\omega)$$

Stochastic Sobolev spaces

Let V be a Hilbert space. We define the stochastic Hilbert spaces $(S)_{\rho,k,V}$ as the set of all formal sums

$$(\mathcal{S})_{
ho,k,V} := \left\{ v = \sum_{lpha \in \mathcal{I}} v_lpha H_lpha \ : \ v_lpha \in V \ \text{ and } \ \|v\|_{
ho,k,V} < \infty
ight\}$$

where $\|\cdot\|_{\rho,k,V}$ denote the norm

$$\begin{split} \|u\|_{\rho,k,V} &:= \left(\sum_{\alpha \in \mathcal{I}} (\alpha!)^{1+\rho} \|u_{\alpha}\|_{V}^{2} (2\mathbb{N})^{k\alpha}\right)^{\frac{1}{2}} \\ (u,v)_{\rho,k,V} &:= \sum_{\alpha \in \mathcal{I}} (u_{\alpha}, v_{\alpha})_{V} (\alpha!)^{1+\rho} (2\mathbb{N})^{k\alpha}, u, v \in (\mathcal{S})_{\rho,k,V} \\ (\mathcal{S})_{\rho,k,V} &\cong \mathcal{L}(V', (\mathcal{S})_{\rho,k}) \cong V \otimes (\mathcal{S})_{\rho,k} \\ &= \sum_{\alpha \in \mathcal{I}} (\mathcal{S})_{\rho,k,V} \cong \mathcal{L}(V', (\mathcal{S})_{\rho,k}) \cong V \otimes (\mathcal{S})_{\rho,k,V} \\ &= \sum_{\alpha \in \mathcal{I}} (\mathcal{S})_{\rho,k,V} \cong \mathcal{L}(V', (\mathcal{S})_{\rho,k}) \cong V \otimes (\mathcal{S})_{\rho,k,V} \\ &= \sum_{\alpha \in \mathcal{I}} (\mathcal{S})_{\rho,k,V} \cong \mathcal{L}(V', (\mathcal{S})_{\rho,k}) \cong V \otimes (\mathcal{S})_{\rho,k,V} \\ &= \sum_{\alpha \in \mathcal{I}} (\mathcal{S})_{\rho,k,V} \cong \mathcal{L}(V', (\mathcal{S})_{\rho,k,V}) \cong V \otimes (\mathcal{S})_{\rho,k,V} \\ &= \sum_{\alpha \in \mathcal{I}} (\mathcal{S})_{\rho,k,V} \cong \mathcal{L}(V', (\mathcal{S})_{\rho,k,V}) \cong V \otimes (\mathcal{S})_{\rho,K,V} \\ &= \sum_{\alpha \in \mathcal{I}} (\mathcal{S})_{\rho,k,V} \cong \mathcal{L}(V', (\mathcal{S})_{\rho,k,V}) \cong V \otimes (\mathcal{S})_{\rho,K,V} \\ &= \sum_{\alpha \in \mathcal{I}} (\mathcal{S})_{\rho,K,V} \otimes \mathcal{L}(V', (\mathcal{S})_{\rho,K,V}) \cong V \otimes (\mathcal{S})_{\rho,K,V} \\ &= \sum_{\alpha \in \mathcal{I}} (\mathcal{S})_{\rho,K,V} \otimes \mathcal{L}(V', (\mathcal{S})_{\rho,K,V}) \cong V \otimes (\mathcal{S})_{\rho,K,V} \\ &= \sum_{\alpha \in \mathcal{I}} (\mathcal{S})_{\rho,K,V} \otimes \mathcal{L}(V', (\mathcal{S})_{\rho,K,V}) \cong V \otimes (\mathcal{S})_{\rho,K,V} \\ &= \sum_{\alpha \in \mathcal{I}} (\mathcal{S})_{\rho,K,V} \otimes \mathcal{L}(V', (\mathcal{S})_{\rho,K,V}) \cong V \otimes (\mathcal{S})_{\rho,K,V} \\ &= \sum_{\alpha \in \mathcal{I}} (\mathcal{S})_{\rho,K,V} \otimes \mathcal{L}(V', (\mathcal{S})_{\rho,K,V}) \cong V \otimes (\mathcal{S})_{\rho,K,V} \\ &= \sum_{\alpha \in \mathcal{I}} (\mathcal{S})_{\rho,K,V} \otimes \mathcal{L}(V', (\mathcal{S})_{\rho,K,V}) \otimes \mathcal{L}(V', (\mathcal{S})_{\rho,K,V}) \\ &= \sum_{\alpha \in \mathcal{I}} (\mathcal{S})_{\rho,K,V} \otimes \mathcal{L}(V', (\mathcal{S})_{\rho,K,V}) \otimes \mathcal{L}(V', (\mathcal{S})_{\rho,K,V})$$

Stochastic Sobolev spaces

• If $\mathcal{D} \subset \mathbb{R}^d$ is bounded and $0 < T < \infty$, then we have:

$$W, W_H \in (S)_{\rho,k,L^2([0,T] \times D)}$$
 for any $-1 \le \rho \le 1$ and $k < 0$
 $W, W_H \in (S)_{-1,l,L^{\infty}([0,T] \times D)}$ for any $l < 0$

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Wick product

• The Wick product $f \diamond g$ of two formal series $f = \sum_{\alpha} f_{\alpha} H_{\alpha}$, $g = \sum_{\alpha} g_{\alpha} H_{\alpha}$ is defined as $f \diamond g := \sum_{\alpha,\beta \in \mathcal{I}} f_{\alpha} g_{\beta} H_{\alpha+\beta}$

Let $\mathcal{D} \subset \mathbb{R}^d$ be open, and let $l \in \mathbb{R}$. We introduce the Banach space

$$\mathcal{F}_{I}(\mathcal{D}) = \{ f = \sum_{lpha \in \mathcal{I}} f_{lpha} \mathcal{H}_{lpha}, f_{lpha} : \mathcal{D} \longrightarrow \mathbb{R} \; \; \mathsf{measurable} \; orall lpha \in \mathcal{I} \}$$

$$\|f\|_{l,*} = \mathrm{ess} \sup_{x \in \mathcal{D}} \left(\sum_{\alpha \in \mathcal{I}} |f_{\alpha}(x)| (2\mathbb{N})^{l\alpha} \right) < \infty$$

Wick product

If $f \in \mathcal{F}_{l}(\mathcal{D})$ and if $g \in \mathcal{S}^{-1,k,L^{2}(\mathcal{D})}$ with $k \leq 2l$, then

$$f \diamond g \in \mathcal{S}_{-1,k,L^2(\mathcal{D})}$$

$$\|f \diamond g\|_{-1,k,0} \le \|f\|_{I,*} \|g\|_{-1,k,0}$$

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Wick exponential

The Wick-exponential of the standard white noise is defined by

$$\begin{split} \exp^{\diamond}(W(t,x)) &= \sum_{\alpha \in \mathcal{I}} \frac{1}{\alpha!} \left(\prod_{i,j=1}^{l(\alpha)} (e_i(t)\eta_j(x))^{\alpha_{ij}} \right) H_{\alpha} \\ \exp^{\diamond}(W_H(t,x)) &= \sum_{\alpha \in \mathcal{I}} \frac{1}{\alpha!} \left(\prod_{i,j=1}^{l(\alpha)} (M_H e_i(t)M_H \eta_j(x))^{\alpha_{ij}} \right) H_{\alpha} \end{split}$$

 $\exp^{\diamond} W, \exp^{\diamond} W_{\mathcal{H}} \in (\mathcal{S})_{-1, l, L^{\infty}([0, T] \times \mathcal{D})} \text{ for } l < 0$

-Four examples

The pressure equation

The pressure equation

We want to solve the following problem:

(1)
$$\begin{cases} \text{Find } p(x,\omega) \text{ solution of the linear SPDE} \\ -\nabla \cdot (\kappa(x,\omega) \diamond \nabla p) = f, & \text{in } \mathcal{D} \times \Omega \\ p(x,\omega) = 0, & \text{on } \partial \mathcal{D} \times \Omega \end{cases}$$

For the flow in a porous medium, p(x, ω) denotes the pressure, κ is the permeability of the medium, f represents the external forces (for example sources or sinks in an oil-reservoir). We allow f and κ to be generalized stochastic distributions, assuming their chaos expansion explicitly known.

Four examples

The pressure equation

► Example 1:

$$\kappa(x,\omega) = \kappa_0(x) + \lambda e^{\diamond W^v(x,\omega)}$$

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The pressure equation

Example 1:

$$\kappa(x,\omega) = \kappa_0(x) + \lambda e^{\diamond W^{\nu}(x,\omega)}$$
Example 2: $\kappa(x,\omega) = \exp(G(x,\omega))$

$$G(x,\omega) = \sum_{m=1}^{\infty} \sqrt{\lambda_m} G_m(x) X_m(\omega)$$

$$\kappa(x,\omega) = \sum_{lpha \in \mathcal{I}} \kappa_{lpha}(x) \mathcal{H}_{lpha}(\omega)$$

$$\kappa_{\alpha}(x) = \frac{\langle \kappa \rangle}{\sqrt{\alpha !}} \prod_{m=1}^{\infty} \left(\sqrt{\lambda_m} G_m(x) \right)^{\alpha_m}$$

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Four examples

The pressure equation

The mixed formulation

$$\begin{cases} u(x,\omega) - \mathcal{K}(x,\omega) \diamond \nabla p(x,\omega) &= 0 & \text{in } \mathcal{D} \times \Omega, \\ -\operatorname{div} u(x,\omega) &= f(x,\omega) & \text{in } \mathcal{D} \times \Omega, \\ p(x,\omega) &= 0 & \text{on } \partial \mathcal{D} \times \Omega \end{cases}$$

$$egin{array}{lll} {a(u,v)} &:= (K^{\diamond(-1)} \diamond u,v)_{-1,k,0}, \ {b(u,q)} &:= (q, {
m div}(u))_{-1,k,0} \end{array}$$

Four examples

The pressure equation

Stochastic Soboloev spaces



Four examples

The pressure equation

The mixed formulation

The mixed variational problem can be written as follows:

$$\left\{ \begin{array}{l} \mathsf{Find} \ u \in \mathcal{H}(\mathsf{div}; \mathcal{D}), p \in \mathcal{L}^2(\mathcal{D}) \text{ such that} \\ a(u, v) + b(v, p) = 0, \quad \forall v \in \mathcal{H}(\mathsf{div}; \mathcal{D}), \\ b(u, q) = (-f, q)_{-1,k,0}, \quad \forall q \in \mathcal{L}^2(\mathcal{D}) \end{array} \right.$$

Suppose that (u, p) ∈ H(div; D) × L²(D) solves the weak formulation and let K^{◊(−1)} be in L[∞]_l(D) for some l such that k ≤ 2l. Then the pressure p is in H⁰_l(D).

Four examples

The pressure equation

The continuity properties

Suppose that K^{◊(−1)} is in L[∞]_I(D) for some I such that k ≤ 2I. Then the bilinear forms a(·, ·) and b(·, ·) are continuous and it holds

$$\begin{aligned} |a(u,v)| &\leq C_a \|u\|_{-1,k,div} \|v\|_{-1,k,div}, \\ |b(v,q)| &\leq C_b \|v\|_{-1,k,div} \|q\|_{-1,k,0} \end{aligned}$$

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for suitable constants $C_a, C_b < \infty$.

-Four examples

The pressure equation

The coercivity property

$$Z = \{v \in \mathcal{H}(\operatorname{div}; \mathcal{D}) : \operatorname{div}(v(x)) = 0 \text{ a.e. } x \in \mathcal{D}\}.$$

Suppose that K^{◊(−1)} is in L[∞]_I(D) for some I such that k ≤ 2I. Then if the parameter k is small enough, the bilinear form a(·, ·) is coercive on Z. That is, it holds

$$a(v,v) \ge \theta_a \|v\|_{-1,k,div}^2$$
 $(\forall v \in Z)$

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for some constant $\theta_a > 0$ and k small enough.

Four examples

The pressure equation

The coercivity property

Since $E[K^{\diamond -1}(x)] = 1/E[K(x)]$ it is clear that the bilinear form $(g, h) \mapsto (E[K^{\diamond (-1)}]g, h)_0$ is coercive on $(L^2(\mathcal{D}))^d$.

$$(K^{\diamond(-1)}\diamond u, v)_{-1,k,0} = \sum_{\gamma} \int_{\mathcal{D}} (\sum_{\alpha+\beta=\gamma} K_{\alpha}^{\diamond(-1)} u_{\beta}) v_{\gamma} dx (2\mathbb{N})^{k\gamma} (K^{\diamond(-1)}\diamond u, v)_{-1,k,0} \ge \sum_{\gamma\in\mathcal{I}} (E[K^{\diamond(-1)}] u_{\gamma}, v_{\gamma})_{0} (2\mathbb{N})^{k\gamma}$$

$$-\frac{1}{2}2^{k/2-l}\|\mathcal{K}^{\diamond(-1)}\|_{l,\infty}(\|u\|_{-1,k,0}^2+\|v\|_{-1,k,0}^2)$$

for each $u, v \in (\mathcal{L}^2(\mathcal{D}))^d$. Thus, for a suitable constant $\theta_0 > 0$, we have

$$(K^{\diamond(-1)}\diamond u, u) \ge (\theta_0 - 2^{k/2-l} \|K^{\diamond(-1)}\|_{l,\infty}) \|u\|_{-1,k,0}^2$$
⁽¹⁾

Choosing the parameter k small enough makes the right-hand side in (1) positive, and since

 $||u||_{-1,k,0} = ||u||_{-1,k,div}$ for all $u \in Z$, the result follows.

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Four examples

The pressure equation

The Inf-sup condition

The bilinear form b(·, ·) satisfies the inf-sup condition: There exists a positive constant θ_b such that

$$\sup_{v\in\mathcal{H}(div),v\neq 0}\frac{b(v,q)}{\|v\|_{-1,k,div}}\geq \theta_b\|q\|_{-1,k,0}, \ \ (\forall q\in\mathcal{L}^2(\mathcal{D}))$$

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The pressure equation

Existence and uniqueness

Suppose given f ∈ L²(D), let K^{◊(-1)} be in L₁[∞](D) for I such that k ≤ 2I, and assume that the parameter k is fixed and small enough. Then the mixed variational formulation has a unique solution (u, p) ∈ H(div; D) × L²(D). Moreover, the following estimates holds

$$\begin{split} \|u\|_{-1,k,\mathrm{div}} &\leq \frac{1}{\theta_b} \big(1 + \frac{C_a}{\theta_a}\big) \|f\|_{-1,k,0} \\ \|p\|_{-1,k,0} &\leq \frac{C_a}{\theta_b^2} \big(1 + \frac{C_a}{\theta_a}\big) \|f\|_{-1,k,0}. \end{split}$$

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Four examples

The pressure equation

The discrete problem

We construct a sequence $\{(\mathcal{X}_m, \mathcal{Q}_m) : m \in \mathbb{N}\}$ of finite dimensional subspaces of $\mathcal{H}(\operatorname{div}, \mathcal{D}) \times \mathcal{L}^2(\mathcal{D})$, and consider the discrete problems

Find
$$u_m \in \mathcal{X}_m, p_m \in \mathcal{Q}_m$$
 such that
 $a(u_m, v) + b(v, p_m) = 0, (\forall v \in \mathcal{X}_m),$
 $b(u_m, q) = (-f, q)_{-1,k,0}, (\forall q \in \mathcal{Q}_m),$

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The pressure equation

Approximate spaces

- ▶ \mathcal{T}_h is a finite collection of open triangles (or tetrahedra) { $T_i : i = 1, ..., r$ } such that $T_i \cap T_j = \emptyset$ if $i \neq j$.
- ▶ For a given domain $T \subset \mathbb{R}^d$, and $n \in \mathbb{N}_0$, we define the spaces

$$\mathbb{D}_n(T) := (\mathbb{P}_{n-1}(T))^d \oplus x \mathbb{P}_{n-1}(T)$$
$$X_h^n := \{ v \in H(\operatorname{div}; \mathcal{D}) : v |_T \in \mathbb{D}_n(T), \ T \in \mathcal{T}_h \}$$
$$Q_h^{n-1} := \{ v \in L^2(\mathcal{D}) : v |_T \in \mathbb{P}_{n-1}(T), \ T \in \mathcal{T}_h \}$$

Four examples

The pressure equation

Approximate spaces

For $N, K \in \mathbb{N}$ we define the cutting $\mathcal{I}_{N,K} \subset \mathcal{I}$ by

$$\mathcal{I}_{N,K} := \{0\} \cup \bigcup_{n=1}^{N} \bigcup_{k=1}^{K} \{\alpha \in \mathbb{N}_{0}^{k} : |\alpha| = n \text{ and } \alpha_{k} \neq 0\}$$

Next, for each $h \in (0, 1]$ and $n, N, K \in \mathbb{N}$ we define the finite-dimensional spaces

$$\mathcal{X}_{N,K,h}^{n} := \{ v = \sum_{lpha \in \mathcal{I}_{N,K}} v_{lpha} H_{lpha} \in \mathcal{H}(\mathsf{div}; \mathcal{D}) : v_{lpha} \in X_{h}^{n} \}$$

 $\mathcal{Q}_{N,K,h}^{n-1} := \{ q = \sum_{lpha \in \mathcal{I}_{N,K}} q_{lpha} H_{lpha} \in \mathcal{L}^{2}(\mathcal{D}) : q_{lpha} \in Q_{h}^{n-1} \}$

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The pressure equation

Approximate spaces

▶ Let $m \in \mathbb{N}$ denote some ordering of the parameters N, K, rsuch that $N^{(m)} + K^{(m)} + r^{(m)} \leq N^{(m+1)} + K^{(m+1)} + r^{(m+1)}$

$$\begin{aligned} \mathcal{X}_{m} &:= \mathcal{X}_{N^{(m)}, K^{(m)}, h_{r^{(m)}}}^{n} \text{ and } \mathcal{Q}_{m} := \mathcal{Q}_{N^{(m)}, K^{(m)}, h_{r^{(m)}}}^{n-1} \\ \triangleright \ \mathcal{X}_{m} \subset \mathcal{H}(\operatorname{div}; \mathcal{D}) \text{ and } \mathcal{Q}_{m} \subset \mathcal{L}^{2}(\mathcal{D}) \ (m \in \mathbb{N}) \\ \triangleright \ \operatorname{div}(\mathcal{X}_{m}) &= \mathcal{Q}_{m} \\ \triangleright \ v^{N, K} := \sum_{\alpha \in \mathcal{I}_{N, K}} v_{\alpha} H_{\alpha}, \quad v_{m} = v^{N^{(m)}, K^{(m)}} \end{aligned}$$

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Four examples

The pressure equation

Discrete coercivity and Inf-sup conditions

$$Z_m = \{ v \in \mathcal{X}_m : b(v,q) = 0, q \in \mathcal{Q}_m \}$$
$$Z_m \subset Z$$
$$a(v,v) \ge \vartheta_a \|v\|_{-1,k,\mathrm{div}}^2, \qquad \forall v \in Z_m$$
$$\sup_{v \in \mathcal{X}_m, v \neq 0} \frac{(q,\mathrm{div}(v))_{-1,k,0}}{\|v\|_{-1,k,\mathrm{div}}} \ge \vartheta_b \|q\|_{-1,k,0}, \ \forall q \in \mathcal{Q}_m$$

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Four examples

The pressure equation

Existence and unicity for the discrete solution

The discrete problem has a unique solution $(u_m, p_m) \in \mathcal{X}_m \times \mathcal{Q}_m$.

Moreover, it holds

$$\begin{aligned} \|u_m\|_{-1,k,div} &\leq \frac{1}{\vartheta_b} \left(1 + \frac{C_a}{\vartheta_a}\right) \|f\|_{-1,k,0} \\ \|p_m\|_{-1,k,0} &\leq \frac{C_a}{\vartheta_b^2} \left(1 + \frac{C_a}{\vartheta_a}\right) \|f\|_{-1,k,0} \end{aligned}$$

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The pressure equation

Error estimates

Let (u, p) and (u_m, p_m) be the solutions of the continuous and the discrete weak problems. If $(u, p) \in (\mathcal{H}^l(\mathcal{D}))^d \times \mathcal{H}^l(\mathcal{D})$, $1 \leq l \leq n$, then it holds

$$||u - u_m||_{-1,k,0} \le ||u - u^{N,K}||_{-1,k,0} + Ch^{I} ||u||_{-1,k,I}$$

$$\|p - p_m\|_{-1,k,0} \le \|p - p^{N,K}\|_{-1,k,0} + Ch^{I}(\|p\|_{-1,k,I} + \|u\|_{-1,k,I})$$

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for a suitable positive constant C, independent of N, K, and h.



The pressure equation

Truncation errors

It remains only to estimate the truncation errors

$$\|u - u^{N,K}\|_{-1,k,0}$$
 and $\|p - p^{N,K}\|_{-1,k,0}$.

Let V be any separable Hilbert space, let $N, K \in \mathbb{N}, q \ge 0$ be given, and assume $r > r^*$, where r^* solves

$$rac{r^*}{2^{r^*}(r^*-1)}=1$$
 ($r^*pprox 1.54$).

Four examples

The pressure equation

Truncation errors

Then for $f \in (\mathcal{S})^{-1,-(q+r),V}$ it holds

$$\|f - f^{N,K}\|_{-1,-(q+r),V} \le B_{N,K} \|f\|_{-1,-q,V}$$

where

$$B_{K,N} = \sqrt{C_1(r)K^{1-r} + C_2(r)(\frac{r}{2^r(r-1)})^{N+1}},$$

$$C_1(r) = \frac{1}{2^r(r-1)-r}, \quad C_2(r) = 2^r(r-1)C_1(r)$$
(Benth, Gjerde, Vage)

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The pressure equation

Main results

Let $q \ge 0$ and assume $r > r^*$. Then if $(u, p) \in (\mathcal{H}^{l}(\mathcal{D}))^d \times \mathcal{H}^{l}(\mathcal{D})$, $1 \le l \le n$, and with the parameter k = -(q + r), it holds

$$||u - u_m||_{-1,k,0} \le B_{N,K} ||u||_{-1,-q,0} + Ch' ||u||_{-1,k,l}$$

$$\|p - p_m\|_{-1,k,0} \le B_{N,K} \|p\|_{-1,-q,0} + Ch'(\|p\|_{-1,k,l}) + \|u\|_{-1,k,l}$$

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for some positive constant C independent of N,k and h.

The pressure equation

Remarks

- Rate of convergence in the spatial dimension is optimal.
- The approximation of the seepage velocity u has the same order of accuracy as that of the pressure p.
- Because the rate in the stochastic dimension, is rather low, a priori, there is little point in using high order elements when constructing X_hⁿ and Q_hⁿ⁻¹ (one could, for example, choose X_h¹ and Q_h⁰).
- In those cases where the solution has high stochastic regularity, the observed stochastic rate seems to be quite fast. In this case using higher order finite elements may be appropriate.

-Four examples

The pressure equation

Algorithmic aspects of the approximation

- We study algorithmic aspects of our approximation.
- We show how the approximation can be constructed as a sequence of deterministic mixed finite element problems, and indicate a suitable approach for the solution of this sequence.

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• We also discuss stochastic simulation of the solution.

Four examples

The pressure equation

Chaos coefficients

Let $v = wH_{\gamma}$ and $q = gH_{\gamma}$ with $w \in H(\text{div}; D)$, $g \in L^{2}(D)$, and $\gamma \in \mathcal{I}$, then

$$\begin{aligned} \mathsf{a}(u,v) &= ((K^{\diamond(-1)}\diamond u)_{\gamma},w)_0(2\mathbb{N})^{k\gamma} \\ &= \sum_{\alpha+\beta=\gamma} (K_{\beta}^{\diamond(-1)}u_{\alpha},w)_0(2\mathbb{N})^{k\gamma} \end{aligned}$$

$$b(u,q)=(q,\operatorname{div}(u))_{-1,k,0}=(g,\operatorname{div}(u_\gamma))_0(2\mathbb{N})^{k\gamma}$$

Four examples

The pressure equation

Chaos coefficients

Thus, the chaos coefficients $\{(u_{m,\gamma}, p_{m,\gamma}) : \gamma \in \mathcal{I}_{N,K}\}$ must solve the following set of variational problems. For each $\gamma \in \mathcal{I}_{N,K}$, find $u_{m,\gamma} \in X_h^n$ and $p_{m,\gamma} \in Q_h^n$, such that:

$$(1) \begin{cases} a_0(u_{m,\gamma},w) + b_0(p_{m,\gamma},w) = -\sum_{\alpha \prec \gamma} a_{\gamma-\alpha}(u_{m,\alpha},w), \\ b_0(g,u_{m,\gamma}) = (-f_{\gamma},g)_0, \\ \forall w \in X_h^n, \, \forall g \in Q_h^n \end{cases}$$

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The pressure equation

Chaos coefficients

where we have introduced the bilinear operators $a_{\beta}(\cdot, \cdot)$ and $b_{0}(\cdot, \cdot)$ defined on $H(\operatorname{div}; \mathcal{D}) \times H(\operatorname{div}; \mathcal{D})$ and $L^{2}(\mathcal{D}) \times H(\operatorname{div}; \mathcal{D})$, respectively, and given by

$$a_eta(v,w) := ({K}^{\diamond(-1)}_eta v,w)_0,$$

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 $b_0(v,g) := (g, \operatorname{div}(v))_0$

-Four examples

The pressure equation

Ordering

Remark 1:

We shall assume that the set of multi-indices $\mathcal{I}_{N,K}$ is ordered in such a way that $\{u_{m,\beta} : \beta \prec \gamma\}$ has been calculated when the γ th equation is considered. This is essential for the practical use, because such an ordering allows us to solve (1) as a sequence of problems, each giving one of the chaos coefficients of the approximation.

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-Four examples

The pressure equation



Remark 2:

The solution of (1) involves solving (N + K)!/(N!K!)sub-problems. One problem for each $\gamma \in \mathcal{I}_{N,K}$. Also note that the γ th equation is equivalent to the discrete version of a deterministic mixed finite element problem over $H(\operatorname{div}, \mathcal{D}) \times L^2(\mathcal{D})$.

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-Four examples

The pressure equation

The algebraic problem

Let $\{\Psi_i : i = 1, ..., M_X\}$ and $\{\phi_i : i = 1, ..., M_Q\}$ denote the finite element basis functions for X_h^n and Q_h^n , respectively. Then for each $\gamma \in \mathcal{I}_{N,K}$ and $x \in \mathcal{D}$, we may write

$$u_{m,\gamma}(x) = \sum_{i=1}^{M_X} U_{m,\gamma,i} \Psi_i(x), \;\; ext{and} \;\;$$

$$p_{m,\gamma}(x) = \sum_{k=1}^{M_Q} P_{m,\gamma,k} \phi_k(x),$$

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for suitable real constants $U_{m,\gamma,i}$ and $P_{m,\gamma,k}$.

Four examples

The pressure equation

The algebraic problem

Furthermore, $U_{m,\gamma} := [U_{m,\gamma,i}]$ and $P_{m,\gamma} := [P_{m,\gamma,k}]$ satisfy the algebraic problem

$$\begin{bmatrix} A_0 & B_0^T \\ B_0 & 0 \end{bmatrix} \begin{bmatrix} U_{m,\gamma} \\ P_{m,\gamma} \end{bmatrix} = \begin{bmatrix} G_\gamma \\ F_\gamma \end{bmatrix},$$

where we have defined

$$\begin{aligned} &A_{\beta,ij} := a_{\beta}(\Psi_j, \Psi_i), \ B_{0,kj} := b_0(\Psi_j, \phi_k) \\ &G_{\gamma} := -\sum_{\alpha \prec \gamma} A_{\gamma-\alpha} U_{m,\alpha}, \ F_{\gamma,k} := (-f_{\gamma}, \phi_k)_0 \\ &(i, j = 1, \dots, M_X, \ k = 1, \dots, M_Q, \ \gamma, \beta \in \mathcal{I}_{N,K}) \end{aligned}$$

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The pressure equation

Stochastic simulation of the solution

Once we have calculated the chaos coefficients $\{(u_{m,\gamma}, p_{m,\gamma}) : \gamma \in \mathcal{I}_{N,K}\}$, we may do stochastic simulations of the solution as follows: First, generate K independent standard Gaussian variables $X(\omega) = (X_i(\omega))$ (i = 1, ..., K) using some random number generator, and then form the sums

$$\begin{split} u_m(x,\omega) &= \sum_{\alpha \in \mathcal{I}_{N,K}} u_{m,\alpha}(x) H_\alpha(X(\omega)), \\ p_m(x,\omega) &= \sum_{\alpha \in \mathcal{I}_{N,K}} p_{m,\alpha}(x) H_\alpha(X(\omega)), \quad (x \in \mathcal{D}) \\ \text{where } H_\alpha(X(\omega)) &:= \prod_{j=1}^K h_{\alpha_j}(X_j(\omega)) \end{split}$$

-Four examples

The pressure equation

Stochastic simulation of the solution

The advantage of this approach is that it enables us to generate random samples easy and fast. For example, in situations where one is interested in repeated simulations of the pressure and velocity, one may compute the chaos coefficients in advance, store them, and produce the simulations whenever they are needed.

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-Four examples

The pressure equation

An algorithm for the solution

(1) Form the ordered set $\mathcal{I}_{N,K}$ and let $\gamma = (0, \dots, 0)$. (2) Calculate the matrices $A_0 = [a_0(\Psi_i, \Psi_i)]$ and $B_0 = [b_0(\Psi_i, \phi_k)].$ (3) While $\gamma \in \mathcal{I}_{N,K}$ do, (3.1) Calculate $F_{m,\gamma} = [(-f_{\gamma}, \phi_k)_{0,D}].$ (3.2) Find the set $\mathcal{L}_{\gamma} = \{ \alpha \in \mathcal{I}_{N,K} : \alpha \prec \gamma \}.$ (3.3) For each $\alpha \in \mathcal{L}_{\gamma}$, (3.3.1) Calculate the matrices $A_{\gamma-\alpha} = [a_{\gamma-\alpha}(\Psi_i, \Psi_i)].$ (3.3.2) Update the right hand side $G_{m,\gamma} := G_{m,\gamma} - A_{\gamma-\gamma} U_{m,\gamma}.$

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The pressure equation

An algorithm for the solution

(3.4) Solve $(B_0 A_0^{-1} B_0^T) P_{m \gamma} = B_0 A_0^{-1} G_{\gamma} - F_{\gamma}$. (3.5) Solve $A_0 U_{m,\gamma} = G_{\gamma} - B_0^T P_{m,\gamma}$. (4) Find the next multi-index γ and go to Step 3. (5) Create a sequence of RK independent Gaussian variables $\{X_i, i = 1, \ldots, RK\}$. (6) For each r = 1, ..., R do, (6.1) Set $X^{(r)} := [X_{(r-1)K+i}]$ (j = 1, ..., K). (6.2) Form simulations of the velocity $u_m^{(r)}(x) = \sum_{\alpha \in \mathcal{I}_{N,K}} u_{m,\alpha}(x) H_{\alpha}(X^{(r)})$ and the pressure $p_m^{(r)}(x) = \sum_{\alpha \in \mathcal{I}_N \kappa} p_{m,\alpha}(x) H_\alpha(X^{(r)}).$

Four examples

The pressure equation

A numerical example

We consider

$$\mathcal{D} =] - 5, +5[$$

$$\mathcal{K}(x) := \exp^{\diamond}(\mathcal{W}(x)) = \sum_{n=0}^{\infty} \frac{\mathcal{W}(x)^{\diamond n}}{n!},$$

$$\mathcal{W}(x) = \sum_{j=1}^{\infty} \eta_j(x) H_{\epsilon_j}$$

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Four examples

The pressure equation

Numerical results: Case A



The first two rows of plots show 6 different simulations of the pressure where f = 1 and (N, K) = (3, 15). The last row displays the simulated velocities corresponding to the pressures in the middle row. The dotted line is the

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-Four examples

The pressure equation

Numerical results: Case A



The first plot displays $\|p_{m,\alpha}\|_{\infty}$ as a function of our ordering of $\mathcal{I}_{N,K}$, and the second plot is the corresponding plot for the velocity. Both plots are for Case A where f = 1 and (N, K) = (3, 15).

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-Four examples

The pressure equation

Numerical results: Case A



This figure shows some typical chaos coefficients of the pressure for the Case A where f = 1 and (N, K) = (3, 15). In particular, counted from left to right, these are the coefficients numbered 1, 13, 59, 201, 274, 387, 431, 611 and

797 in our ordering of $\mathcal{I}_{N,K}$.

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-Four examples

The pressure equation

Numerical results: Case B

Case B: Here we assume (N, K) = (1, 816) and set f = 1. Thus, the approximated solution uses the same number of chaos coefficients as in Case A, but with a different set $\mathcal{I}_{N,K}$. Thus, the approximated solution uses the same number of chaos coefficients as in Case A, but with a different set $\mathcal{I}_{N,K}$. We can see from Figure that this leads more irregular behavior of the approximation, in particular, for the pressure. This behavior is a result of the shape and size of the chaos coefficients.

Four examples

The pressure equation

Numerical results: Case B



This is the same type of plot as in Figure (1), now for Case B where f(x) = 1 and (N, K) = (1, 816).

-Four examples

The pressure equation

Numerical results: Case B



The first plot shows $\|p_{m,\alpha}\|_{\infty}$ as a function of our ordering of $\mathcal{I}_{N,K}$ and the second plot is the corresponding plot for the velocity. Both plots are for Case B where f = 1 and (N, K) = (1, 816).

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Four examples

The pressure equation

Case C

In this case we assume (N, K) = (1, 816) and set f(x) = 1 + W(x) $(x \in [-5, 5])$, where W(x) denotes singular white noise. Due to this stochastic forcing, the solution should behave more irregular than in Case B.

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-Four examples

The pressure equation

Numerical results: Case C



f(x) = 1 + W(x) and (N, K) = (1, 816).

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-Four examples

The pressure equation

Numerical results: Case C



The first plot displays $\|p_{m,\alpha}\|_{\infty}$ as a function of α (using our ordering of $\mathcal{I}_{N,K}$), and the second plot is the corresponding plot for the velocity. Both plots are for Case C where f(x) = 1 + W(x) and (N, K) = (1, 816).

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-Four examples

The pressure equation

Numerical results: Case C



The first two lines of plots display the difference $(p_{\alpha}^{B} - p_{\alpha}^{C})(x)$ for some chaos coefficients of the pressure in Cases B and C. In particular, counted from left to right, these are the differences for the coefficients numbered 2, 9, 10, 15, 21, and 28 in our ordering of $\mathcal{I}_{N,K}$. The last row shows the corresponding difference in the velocity, for the coefficients numbered 15, 21, and 28 $(D^{-1})^{-1} \leq D^{-1} < D^{-1} \leq D^{-1} < D^{-1} < D^{-1} < D^{-1}$

Four examples

The pressure equation

Numerical results: Case C



The first plot displays the difference $\|p_{m,\alpha}^{B} - p_{\alpha}^{C}\|_{\infty}$ as a function of α (using our ordering of $\mathcal{I}_{N,K}$). The second plot is the corresponding difference for the velocity u.

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Four examples

Linear SPDE with additive and multiplicative noises

SPDE with additive and multiplicative noises

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla \cdot (\kappa \diamond \nabla u) - ru \diamond W_1(t, \omega) = f + \sigma W_2(t, x, \omega) & \text{ in } (0, T) \times T \\ u(t, x, \omega) = 0 & \text{ on } [0, T] \times \\ u(0, x, \omega) = g(x, \omega) & \text{ in } \mathcal{D} \times \Omega \\ \end{cases}$$
(2)

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We consider the two-dimensional stationary case of (1) with permeability $\kappa = 1$. We will in this example consider two specific cases: Cases A and B, corresponding to the additive and multiplicative white noises respectively.

Four examples

Linear SPDE with additive and multiplicative noises

Case A:

We assume (N, K) = (3, 3) and set f = 1, r = 0 and $\sigma = 1$. In Figure 1 we show typical simulations for the pressure. We also plot some of the chaos coefficients and some realizations of the solution.

Mean and Chaos coefficient 1



Linear SPDE with additive and multiplicative noises

Chaos coefficients 2 and 4



Four examples

Linear SPDE with additive and multiplicative noises

Realizations 1 and 8



Four examples

Linear SPDE with additive and multiplicative noises

Realizations 12 and 20



Four examples

Linear SPDE with additive and multiplicative noises

Case B:

We assume (N, K) = (3, 3) and set f = 1, r = 1 and $\sigma = 0$. In Figure 2, we show typical simulations for the pressure. We also plot some of the chaos coefficients and some realizations of the solution.

Mean and Chaos coefficient 1



Linear SPDE with additive and multiplicative noises

Chaos coefficients 2 and 4



Four examples

Linear SPDE with additive and multiplicative noises

Realizations 1 and 8



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Linear SPDE with additive and multiplicative noises

Realizations 12 and 20



Helmholtz equation with stochastic refractive index

Stochastic micro-structured photonic crystal fibers

Photonic crystal fibers (PCF) consist of an array of holes running through the length of the fibers which serve as cores for light guiding.

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Helmholtz equation with stochastic refractive index

The governing equation for PCF is the Maxwell equation. By ssuming time harmonic $e^{-i\omega t}$ and z dependence $e^{i\beta z}$ along the fiber the Maxwell can be reduced to a Helmholtz equation with unknown complex propagation constant β .

$$\Delta E + k^2 E = 0$$

- $k^2 = k_0^2 n^2 \beta^2$. • $k_0 = \frac{2\pi}{\lambda}$ = the wave number. • $n = n(\lambda)$ refractive index.
- $\beta = \text{effective index.}$

$$e^{i\beta z} = e^{i(\operatorname{\mathsf{Re}}(\beta))z}e^{-\operatorname{\mathsf{Im}}(\beta)z}$$

 $\operatorname{Re}(\beta)$ gives the propagation constant of the light along the fiber and $\operatorname{Im}(\beta)$ gives the decay rates.

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Helmholtz equation with stochastic refractive index



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Helmholtz equation with stochastic refractive index



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Helmholtz equation with stochastic refractive index



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Helmholtz equation with stochastic refractive index



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Helmholtz equation with stochastic refractive index


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Stochastic schallow water equations

Schallow water: the problem

The problem



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Stochastic schallow water equations

Schallow water: notations

- $B = B(x, y, \omega) =$ topography variations
- $D = D(t, x, y, \omega) =$ total length of the water column
- φ = φ(t, x, y, ω) = local water elevation from the surface
 z=0.φ = B + D.



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Stochastic schallow water equations

Schallow water: equations

$$\frac{\partial \overrightarrow{u}}{\partial t} + \overrightarrow{u} \nabla \overrightarrow{u} - \nu \Delta \overrightarrow{u} = -f \overrightarrow{u}^{\perp} - g \nabla \phi$$
$$\frac{\partial \phi}{\partial t} + \overrightarrow{u} \cdot \nabla \phi + \phi \nabla \overrightarrow{u} = \overrightarrow{u} \cdot \nabla B + B \nabla \overrightarrow{u}$$

•
$$\overrightarrow{u} = \overrightarrow{u}(t, x, y, \omega) =$$
 the velocity field, $\overrightarrow{u}^{\perp} = (-u_2, u_1).$

- g = gravitational acceleration
- $(t, x, y) \in [0, T] \times D =$ time-spatial domain
- $\omega \in \Omega$ = set of elementary events

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Stochastic schallow water equations

Uncertain topography

 $B(x, y, \omega) = B_0(x, y) + \tilde{B}(x, y, \omega)$

$$B(x, y, \omega) = \exp^{\diamond}(W(x, y, \omega))$$

Exponential White Noise



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Stochastic schallow water equations

Uncertain topography



bathymetry: realization1

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Stochastic schallow water equations

Uncertain topography



bathymetry: realization2

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Stochastic schallow water equations

Uncertain topography



bathymetry: realization3

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Stochastic schallow water equations

Uncertain topography



bathymetry: realization4

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Stochastic schallow water equations

Uncertain topography



bathymetry: mean

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Stochastic schallow water equations

Chaos coefficients:

Substituting the Wiener chaos expansions

$$\overrightarrow{u}(t,x,y,\omega) = \sum_{\alpha} \overrightarrow{u}_{\alpha}(t,x,y) H_{\alpha}(\omega)$$

$$\phi(t, x, y, \omega) = \sum_{lpha} \phi_{lpha}(t, x, y) \mathcal{H}_{lpha}(\omega)$$

$$B(x,y,\omega) = \sum_{lpha^*} B_lpha(x,y) H_{lpha^*}(\omega), \qquad lpha^* = (lpha_{1j})$$

$$f(t, x, \omega) = \sum_{\alpha} f_{\alpha}(x) H_{\alpha}(\omega)$$

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Stochastic schallow water equations

- we obtain the following recursive system of deterministic PDE's:
- If $\gamma = 0$, then $(\overrightarrow{u}_0, \phi_0)$ is solution of the shallow water equations

$$\frac{\partial \overrightarrow{u}_{0}}{\partial t} + \overrightarrow{u}_{0} \cdot \nabla \overrightarrow{u}_{0} - \nu \Delta \overrightarrow{u}_{0} = f_{0} \overrightarrow{u}_{0}^{\perp} - g \nabla \phi_{0}$$

$$\frac{\partial \phi_0}{\partial t} + \overrightarrow{u}_0 \cdot \nabla \phi_0 + \phi_0 \nabla \cdot \overrightarrow{u}_0 = \overrightarrow{u}_0 \cdot \nabla B_0 + B_0 \nabla \cdot \overrightarrow{u}_0,$$

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Stochastic schallow water equations

• If $\gamma \succ 0$, then $(\overrightarrow{u}_{\gamma}, \phi_{\gamma})$ is solution of the linearized shallow water equations

$$\frac{\partial \overrightarrow{u}_{\gamma}}{\partial t} + \overrightarrow{u}_{0} \cdot \nabla \overrightarrow{u}_{\gamma} + \overrightarrow{u}_{\gamma} \cdot \nabla \overrightarrow{u}_{0} - \nu \Delta \overrightarrow{u}_{\gamma} = -f_{0} \overrightarrow{u}_{\gamma}^{\perp} - g \nabla \phi_{\gamma}$$

$$-f_{\gamma}\overrightarrow{u}_{0}^{\perp}-\sum_{\alpha<\gamma}\overrightarrow{u}_{\alpha}\cdot\nabla\overrightarrow{u}_{\gamma-\alpha}-\sum_{\alpha<\gamma}f_{\alpha}\overrightarrow{u}_{\gamma-\alpha}^{\perp}$$

$$\frac{\partial \phi_{\gamma}}{\partial t} + \overrightarrow{u}_{0} \cdot \nabla \phi_{\gamma} + \phi_{\gamma} \cdot \nabla \overrightarrow{u}_{0} = -\overrightarrow{u}_{\gamma} \cdot \nabla \phi_{0} - \phi_{0} \cdot \nabla \overrightarrow{u}_{\gamma}$$

$$\begin{split} &-\sum_{\alpha<\gamma}\overrightarrow{u}_{\gamma}\nabla\phi_{\gamma-\alpha}-\sum_{\alpha<\gamma}\phi_{\alpha}\nabla.\overrightarrow{u}_{\gamma-\alpha}\\ &+\sum_{\alpha\leq\gamma}\overrightarrow{u}_{\alpha}\nabla B_{\gamma-\alpha}-\sum_{\alpha\leq\gamma}B_{\alpha}\nabla.\overrightarrow{u}_{\gamma-\alpha} \end{split}$$

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Stochastic schallow water equations

Numerical simulations



Four examples

Stochastic schallow water equations



Figure: height deterministic (2d)

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Stochastic schallow water equations



Figure: height stochastic (2d)

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Stochastic schallow water equations



Figure: height deterministic (3d)

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Stochastic schallow water equations



Figure: height stochastic (3d)

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Stochastic schallow water equations



Figure: velocity deterministic

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Stochastic schallow water equations



Figure: velocity stochastic

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Conclusions

- A particular nice feature of the Wick approach is that the singular white noise process can be defined as a mathematical rigorous object.
- SPDEs can be solved as actual PDEs and not only as integral equations.
- Many multiplicative or non-linear SPDEs are well defined in their Wick version.
- SPDEs involving additive and multiplicative noise can be solved numerically.
- We can handle SPDEs involving white noise with dependent increments.

Conclusions

Conclusions

- Wick type SPDEs are easy to solve.
- Ability to handle PDE's with stochastic effects (stochastic boundary, initial conditions, boundary conditions, forces and coefficients which satisfy SDE's, ···).
- Can be used as a preconditioner for more general problems.

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Conclusions

THANK YOU

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