KTH Royal Institute of Technology

# Richardson extrapolation and finite difference schemes for SPDEs

*Center for Uncertainty Quantification Seminar*
KAUST · 5 March 2014

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# Orientation

- We are interested in the strong solution to linear parabolic SPDEs arising in the nonlinear filtering theory
- Solutions are desired in real-time and the non-degeneracy of the parabolicity condition cannot be guaranteed
- Richardson's method has been used for SDEs and more recently for SPDEs to accelerate strong rate of convergence for a semi-discrete scheme [Gyöngy & Krylov 2010]
- Give sufficient conditions for accelerating the strong rate of convergence with respect to the spatial approximation to arbitrarily high order by Richardson's method (strong parabolicity [E.H. 2012] and degenerate parabolicity [E.H. 2013])

# Outline

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# The setting

- Consider a continuous problem with *true solution* $\mathrm{u}$
- Apply a *discretization process* with a *mesh size* $\mathrm{h}$
- Obtain solution $w(\mathrm{h})$ to the resulting discretized problem

# Higher order accuracy

- Assume $w(h) - u = O(h)$ as $h \to 0^+$, that is, first order accuracy
- Halving the mesh size, one (hopes) to obtain an approximation $w(h/2)$ that is twice as accurate as $w(h)$
- Refining the mesh size, one can theoretically improve the approximation to any order of accuracy
- However, decreasing mesh size to small values of $h$ is not realistic

# Richardson's observation

- Richardson (1911) observed that for a particular continuous problem a symmetric finite difference scheme lead to errors of form

$$w(h) - u = u^{(1)}h^2 + u^{(2)}h^4 + u^{(3)}h^6 + \cdots$$

  where $w(0) = u$ and the $u^{(i)}$ are independent of $h$
- Note: $w(h) - u = O(h^2)$, as $h \to 0^+$
- Eliminate the $h^2$ term by combining approximations obtained at two different mesh widths, $h_0$, $h_1$:

$$w(h_0, h_1) := \frac{h_1^2 w(h_0) - h_0^2 w(h_1)}{h_1^2 - h_0^2}$$

# Extrapolation: a simple example

$$w(h_0, h_1) := \frac{h_1^2 w(h_0) - h_0^2 w(h_1)}{h_1^2 - h_0^2}$$

For instance, take $h_0 := h$ and $h_1 := 2h$ for Richardson's problem

$$\begin{array}{rcccccc} 4h^2 w(h) & = & 4h^2 u & + & 4h^4 u^{(1)} & + & 4h^6 u^{(2)} + \cdots \\ -(\ h^2 w(2h) & = & h^2 u & + & 4h^4 u^{(1)} & + & 16h^6 u^{(2)} + \cdots) \\ 3h^2 w(h, 2h) & = & 3h^2 u & + & 0 & + & O(h^6) \end{array}$$

- Thus higher order convergence: $w(h, 2h) - u = O(h^4)$
- Can be repeated to arbitrary order: $w(h_0, \ldots, h_k) - u = O(h^k)$
- Doesn't require access to 'the code'
- Naive method would require quite a mesh refinement…

# Outline

# The SPDE

Let $(W^\rho)_{\rho=0}^{d_1}$ be a sequence of Wiener processes. Consider the Cauchy problem for

$$\begin{aligned} du(t,x) =& (\mathcal{L}u(t,x) + f(t,x))dt \\ &+ \sum_{\rho=1}^{d_1} (\mathcal{M}^\rho u(t,x) + g^\rho(t,x))dW^\rho(t) \end{aligned} \quad \text{(Eq)}$$

on $H_T := [0,T] \times \mathbf{R}^d$ with initial condition $u_0 = u(0,x)$

- $\mathcal{L}(t) := a^{\alpha\beta}(t,x)D_\alpha D_\beta, \quad a^{\alpha\beta} = a^{\beta\alpha}$
- $\mathcal{M}^\rho(t) := b^{\alpha\rho}(t,x)D_\alpha$

for $\alpha, \beta \in \{0, \dots, d\}$.

# Well-posedness

The behavior of

$$\begin{aligned} du(t,x) =&(a^{\alpha\beta}D_\alpha D_\beta u(t,x) + f(t,x))dt \\ &+ \sum_{\rho=1}^{d_1}(b^{\alpha\rho}D_\alpha u(t,x) + g^\rho(t,x))dW^\rho(t) \end{aligned} \tag{Eq}$$

is governed by the quadratic form:

$$(2a^{\alpha\beta} - b^{\alpha\rho}b^{\beta\rho})z_\alpha z_\beta$$

# Well-posedness

The behavior of

$$du(t,x) = (a^{\alpha\beta} D_\alpha D_\beta u(t,x) + f(t,x))dt + \sum_{\rho=1}^{d_1} (b^{\alpha\rho} D_\alpha u(t,x) + g^\rho(t,x)) dW^\rho(t) \qquad (Eq)$$

is governed by the quadratic form:

$$(2a^{\alpha\beta} - b^{\alpha\rho} b^{\beta\rho}) z_\alpha z_\beta \geqslant \kappa |z|^2$$

for $\kappa > 0$ then strongly parabolic.

# Well-posedness

The behavior of

$$\begin{aligned} du(t,x) =& (a^{\alpha\beta} D_\alpha D_\beta u(t,x) + f(t,x))dt \\ &+ \sum_{\rho=1}^{d_1} (b^{\alpha\rho} D_\alpha u(t,x) + g^\rho(t,x)) dW^\rho(t) \end{aligned} \tag{Eq}$$

is governed by the quadratic form:

$$(2a^{\alpha\beta} - b^{\alpha\rho} b^{\beta\rho}) z_\alpha z_\beta \geqslant 0$$

then degenerate parabolic.

# Implicit Euler approximation

Together with $(Eq)$ we consider for each fixed $\tau \in (0, 1)$

$$v_i(x) = v_{i-1}(x) + (\mathcal{L}_i v_i(x) + f_i(x))\tau + \sum_{\rho=1}^{d_1} (\mathcal{M}^\rho_{i-1} v_{i-1}(x) + g^\rho_{i-1}(x))\xi^\rho_i \qquad (Eq_\tau)$$

for $i \in \{1, \dots, n\}$ and $x \in \mathbf{R}^d$ with initial condition $v_0(x) = u_0$ where $\xi^\rho_i := \Delta W^\rho(t_{i-1}) = W^\rho(t_i) - W^\rho(t_{i-1})$.
Note: a continuous problem in $x$.

# Discretizing in space

For $h \in \mathbf{R}\backslash\{0\}$ and finite subset $\Lambda \subset \mathbf{R}^d$ containing the origin, define the space-grid

$$G_h := \left\{\lambda_1 h + \dots + \lambda_p h; \lambda_i \in \Lambda \cup (-\Lambda), p = 1, 2, \dots\right\}$$

and spatial differences

$$\delta_{h,\lambda}\varphi(x) := \frac{\varphi(x + h\lambda) - \varphi(x)}{h}$$

for $\lambda \in \Lambda_0 := \Lambda\backslash\{0\}$, and $\delta_{h,0} := I$.

# Space-time difference approximation

Together with (Eq) and ($Eq_\tau$) we consider for each fixed $\tau$

$$w_i(x) = w_{i-1}(x) + (L_i^h w_i(x) + f_i(x))\tau + \sum_{\rho=1}^{d_1} (M_{i-1}^{h\rho} w_{i-1}(x) + g_{i-1}^\rho(x))\xi_i^\rho \qquad (Eq_\tau^h)$$

for $i \in \{1, \ldots, n\}$ and $x \in G_h$ with a given initial condition.
Here $w_i(x) = w(h, t_i, x)$.

- $L_i^h := \mathfrak{a}_i^{\lambda\mu}(x)\delta_{h,\lambda}\delta_{-h,\mu}, \quad \mathfrak{a}^{\lambda\mu} = \mathfrak{a}^{\mu\lambda}$
- $M_i^{h\rho} := \mathfrak{b}_i^{\lambda\rho}(x)\delta_{h,\lambda}$

for $\lambda, \mu \in \Lambda$.

# Choice of $\Lambda$, $\mathfrak{a}$, $\mathfrak{b}$: Example 1

Let $\Lambda = \{e_0, e_1, \dots, e_d\}$ where $e_0 = 0$ and $e_\alpha$ is $\alpha$th basis vector in $\mathbf{R}^d$ and let $\mathfrak{a}_i^{e_\alpha, e_\beta} = a_i^{\alpha\beta}$ and $\mathfrak{b}_i^{e_\alpha, \rho} = b_i^{\alpha\rho}$ for $\alpha, \beta \in \{0, 1, \dots, d\}$. Then first order derivatives in $(Eq)$ are approximated by usual first order finite differences and

$$\sum_{\lambda, \mu \in \Lambda_0} \mathfrak{a}_i^{\lambda\mu} \delta_{h,\lambda} \delta_{-h,\mu} u = -\mathfrak{a}_i^{\alpha\beta} \delta_{h,e_\alpha} \delta_{h,-e_\beta}$$

which is a standard finite-difference approximation for $a_i^{\alpha\beta} D_\alpha D_\beta u$.

# Choice of $\Lambda$, $\mathfrak{a}$, $\mathfrak{b}$: Example 2

Let $\Lambda_0 = \{\pm e_1, \dots, \pm e_d\}$ and for $\lambda, \mu \in \Lambda_0$ define $\mathfrak{a}_i^{\lambda\mu}$ and $\mathfrak{b}_i^{\lambda\rho}$ by

$$\mathfrak{a}_i^{\pm e_\alpha, \pm e_\beta} = \frac{1}{2} a_i^{\alpha\beta}, \qquad \mathfrak{a}_i^{\pm e_\alpha, \mp e_\beta} = 0, \qquad \mathfrak{b}_i^{\pm e_\alpha, \rho} = \pm \frac{1}{2} b_i^{\alpha\rho},$$

$$\mathfrak{a}_i^{0, \pm e_\alpha} = \mathfrak{a}_i^{\pm e_\alpha, 0} = \pm \frac{1}{4} (a_i^{0\alpha} + a_i^{\alpha 0}),$$

$$\mathfrak{a}_i^{00} = a_i^{00}, \quad \mathfrak{b}_i^{0\rho} = b_i^{0\rho}.$$

This choice corresponds to using symmetric finite differences to approximate the first-order derivatives, e.g. :

$$\sum_{\lambda \in \Lambda_0} \mathfrak{b}_i^{\lambda\rho} \delta_{h,\lambda} u(x) = \sum_{\alpha=1}^{d} b_i^{\alpha\rho} \frac{u(x + h e_\alpha) - u(x - h e_\alpha)}{2h}.$$

# Summary of assumptions

We require:

- consistency condition
- stochastic parabolicity condition
- suitable regularity of $u_0$, $f$, and $g$
- suitable regularity of $a$, $b$, $\mathfrak{a}$, and $\mathfrak{b}$

# The assumptions I

## Assumption (consistency)

*For* $i \in \{0, \dots, n\}$ $\quad \mathfrak{a}_i^{00} = a_i^{00}$,
$\sum_{\lambda\in\Lambda_0} \mathfrak{a}_i^{\lambda 0}\lambda^\alpha + \sum_{\mu\in\Lambda_0} \mathfrak{a}_i^{0\mu}\mu^\alpha = a_i^{\alpha 0} + a_i^{0\alpha}$,
$\sum_{\lambda,\mu\in\Lambda_0} \mathfrak{a}_i^{\lambda\mu}\lambda^\alpha\mu^\beta = a_i^{\alpha\beta}$, $\mathfrak{b}_i^{0\rho} = b_i^{0\rho}$, *and* $\sum_{\lambda\in\Lambda_0} \mathfrak{b}_i^{\lambda\rho}\lambda^\alpha = b_i^{\alpha\rho}$
*for all* $\alpha, \beta \in \{1, \dots, d\}$ *and* $\rho \in \{1, \dots, d_1\}$.

## Assumption (parabolicity)

*There exists* $\kappa > 0$ *such that*
$\sum_{\alpha,\beta=0}^{d}(2a^{\alpha\beta} - b^{\alpha\rho}b^{\beta\rho})z^\alpha z^\beta \geqslant \kappa|z|^2$ *and*
$\sum_{\lambda,\mu\in\Lambda_0}(2\mathfrak{a}^{\lambda\mu} - \mathfrak{b}^{\lambda\rho}b^{\mu\rho})z_\lambda z_\mu \geqslant \kappa\sum_{\lambda\in\Lambda_0}|z|^2$.

# The assumptions II

Assumption (regularity initial condition, free terms)

$u_0 \in L^2(\Omega, \mathcal{F}_0, W_2^{m+1})$, $f \in W_2^m$, $g^\rho \in W_2^{m+1}$ *and*
$\mathcal{K}_m^2 := \mathbb{E} \int_0^T (\|f(t)\|_m^2 + \|g(t)\|_{m+1}^2)\, dt + \mathbb{E}\|u_0\|_{m+1}^2 < \infty$.

Assumption (regularity coefficients)

*The* $a^{\alpha\beta}$, $\mathfrak{a}^{\alpha\beta}$ *and* $b^\alpha$, $\mathfrak{b}^\alpha$ *and their derivatives are, respectively,* $m$ *times and* $m+1$ *times continuously differentiable in* $x$ *and bounded by* $K$.

# Outline

# Summary of assumptions

We require:

- consistency condition;
- stochastic parabolicity condition;
- suitable regularity of $u_0$, $f$, and $g$;
- suitable regularity of $a$, $b$, $\mathfrak{a}$, and $\mathfrak{b}$.

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## Assumption (consistency)

*For* $i \in \{0, \dots, n\}$ $\quad \mathfrak{a}_i^{00} = a_i^{00}$,
$\sum_{\lambda \in \Lambda_0} \mathfrak{a}_i^{\lambda 0} \lambda^\alpha + \sum_{\mu \in \Lambda_0} \mathfrak{a}_i^{0\mu} \mu^\alpha = a_i^{\alpha 0} + a_i^{0\alpha}$,
$\sum_{\lambda, \mu \in \Lambda_0} \mathfrak{a}_i^{\lambda\mu} \lambda^\alpha \mu^\beta = a_i^{\alpha\beta}$, $\mathfrak{b}_i^{0\rho} = b_i^{0\rho}$, *and* $\sum_{\lambda \in \Lambda_0} \mathfrak{b}_i^{\lambda\rho} \lambda^\alpha = b_i^{\alpha\rho}$
*for all* $\alpha, \beta \in \{1, \dots, d\}$ *and* $\rho \in \{1, \dots, d_1\}$.

## Assumption (stoch. parabolicity)

*There exists* $\kappa > 0$ *such that*
$\sum_{\alpha,\beta=0}^{d} (2a^{\alpha\beta} - b^{\alpha\rho} b^{\beta\rho}) z^\alpha z^\beta \geqslant \kappa |z|^2$ *and*
$\sum_{\lambda,\mu \in \Lambda_0} (2\mathfrak{a}^{\lambda\mu} - \mathfrak{b}^{\lambda\rho} b^{\mu\rho}) z_\lambda z_\mu \geqslant \kappa \sum_{\lambda \in \Lambda_0} |z|^2$.

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$u_0 \in L^2(\Omega, \mathcal{F}_0, W_2^{m+1})$, $f \in W_2^m$, $g^\rho \in W_2^{m+1}$ *and*
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Assumption (regularity coefficients)

*The* $a^{\alpha\beta}$, $\mathfrak{a}^{\alpha\beta}$ *and* $b^\alpha$, $\mathfrak{b}^\alpha$ *and their derivatives are, respectively,* $m$ *times and* $m+1$ *times continuously differentiable in* $x$ *and bounded by* $K$.

# The expansion

We would like that a.s.

$$w_i(x) = v_i(x) + \sum_{j=1}^{k} \frac{h^j}{j!} v_i^{(j)}(x) + R_i^{\tau,h}(x) \qquad (A)$$

holds for $i \in \{1, \ldots, n\}$ and $x \in G_h$ where the $v^{(j)}$ are independent of $h$ and

$$\mathbb{E} \max_{i \leqslant n} \sup_{x \in G_h} \left| R_i^{\tau,h}(x) \right|^2 \leqslant N|h|^{2(k+1)} \mathcal{K}_m^2 \qquad (B)$$

for $\mathcal{K}_m^2 := \mathbb{E}\, \|u_0\|_{m+1}^2 + \mathbb{E}\tau \sum_{i=0}^{n} \left( \|f_i\|_m^2 + \|g_i\|_{m+1}^2 \right) < \infty$ and a constant $N$ *independent* of $\tau$ and $h$.

# The expansion result

**Theorem (Expansion)**

*If the assumptions hold with* $m > k + 1 + d/2$ *for* $k \geqslant 0$ *then the error admits an expansion* (A) *with bound* (B) *for the remainder with a constant* $N = N(d, d_1, m, \Lambda, K, \kappa, T)$.

# Acceleration notation

Fix an integer $k \geqslant 0$ and let

$$\bar{w}(h) := \sum_{j=0}^{k} \beta_j w(2^{-j} h)$$

where $w(2^{-j}h) = w(2^{-j}h, t_i, x)$ solves the space-time scheme $(Eq_{\tau}^{h})$ with $2^{-j}h$ in place of $h$. Here $\beta$ is given by $(\beta_0, \beta_1, \dots, \beta_k) := (1, 0, \dots, 0)V^{-1}$ where $V^{-1}$ is the inverse of the Vandermonde matrix with entries $V^{ij} := 2^{-(i-1)(j-1)}$ for $i, j \in \{1, \dots, k+1\}$.

# The acceleration result

Theorem (Acceleration)

*Under the assumptions of the Expansion Theorem,*

$$\mathbb{E}\max_{i\leqslant n}\sup_{x\in G_h}|\bar{w}_i(x)-v_i(x)|^2 \leqslant N|h|^{2(k+1)}\mathcal{K}_m^2$$

*for a constant* $N = N(d, d_1, m, \Lambda, K, \kappa, T)$.

# Proof of acceleration result I

From the expansion, we have that for each $j \in \{0, 1, \dots, k\}$ the terms on the right hand side of $\bar{w}$:

$$w(2^{-j}h) = v + \sum_{i=1}^{k} \frac{h^i}{i!\,2^{ij}} v^{(i)} + \tilde{R}h^{k+1}$$

where $\tilde{R} := h^{-(k+1)} R^{\tau, 2^{-j}h}$.

# Proof of acceleration result II

Then

$$\bar{w} = \left(\sum_{j=0}^{k} \beta_j\right) v + \sum_{j=0}^{k} \sum_{i=1}^{k} \frac{\beta_j h^i}{i!\, 2^{ij}} v^{(i)} + \sum_{j=0}^{k} \beta_j \tilde{R} h^{k+1}$$

$$= v + \sum_{i=1}^{k} \frac{h^i}{i!} v^{(i)} \sum_{j=0}^{k} \frac{\beta_j}{2^{ij}} + \sum_{j=0}^{k} \beta_j \tilde{R} h^{k+1}$$

$$= v + \sum_{j=0}^{k} \beta_j \tilde{R} h^{k+1}$$

since $\sum_{j=0}^{k} \beta_j = 1$ and $\sum_{j=0}^{k} \beta_j 2^{-ij} = 0$ for each $i \in \{1, 2, \ldots, k\}$ by the definition of $(\beta_0, \ldots, \beta_k)$. Then use bound on $R^{\tau,h}$ from expansion Theorem. □

# Proof of expansion result (sketch)

- Although solutions to $(\mathrm{Eq}_\tau^h)$ are *naturally* understood as $\ell^2$-valued functions on $G_h$, it will be advantageous to consider the scheme on the whole space and seek $L^2$-valued solutions
- Obtain estimates in appropriate Sobolev spaces for the ($L^2$-valued) solution of the space-time scheme and time scheme
- Taylor expansion with respect to the spatial mesh is constructed by formally taking derivative (with respect to $h$) of the space-time scheme (results in a system of time discretized SPDEs)
- Use Sobolev embedding and show that (under suitable regularity) the continuous modification restricted to the grid agrees almost surely with the natural definition of the approximate solution

# Proof of expansion result (flavor of estimates)

- For $f, g \in W_2^m$ such that $\mathbb{E}\tau \sum_{i=1}^n (\|f\|_m^2 + \|g\|_m^2) < \infty$ then for each $h \in \mathbf{R} \backslash \{0\}$ equation ($Eq_\tau^h$) admits a unique $W_2^m$ valued solution for any $u_0 \in W_2^{m+1}$ and moreover

$$\mathbb{E}max_{i \leqslant n} \|w_i\|_m^2 + \mathbb{E}\tau \sum_{i=1}^n \sum_{\lambda \in \Lambda} \|\delta_{h,\lambda} w_i\|_m^2 \leqslant N\mathbb{E}\tau(\|u_0\|_{m+1}^2 + \sum_{i=0}^n (\|f_i\|_m^2 + \|g_i\|_m^2))$$

for constant $N = N(d, d_1, m, \Lambda, K, \kappa, T)$

# Proof of expansion result (Taylor expansion) I

- Define operators $\mathcal{L}^{(p)}$ and $\mathcal{M}^{(p)\rho}$ by formally taking the $p$th derivatives in $h$ of $L^h$ and $M^{h\rho}$ at $h = 0$
- Consider a system of time discretized equations

$$\begin{aligned} v_i^{(p)} &= v_{i-1}^{(p)} + (\mathcal{L}_i v_i^{(p)} + \sum_{l=1}^{p} C_p^l \mathcal{L}_i^{(l)} v_i^{(p-l)})\tau \\ &+ (\mathcal{M}_{i-1}^{\rho} v_{i-1}^{(p)} + \sum_{l=1}^{p} C_p^l \mathcal{M}_{i-1}^{(l)\rho} v_{i-1}^{(p-l)})\xi_i^{\rho} \end{aligned} \qquad (\mathcal{S}ys)$$

for $p \in \{1, \ldots, k\}$ with zero initial conditions where $v^{(0)} := \nu$

- Also obtain existence and uniqueness of solutions to $(\mathcal{S}ys)$ and similar estimates *independent of* $\tau$

# Proof of expansion result (Taylor expansion) II

- Now consider $r^{\tau,h} = w - v - \sum_{j=1}^{k} \frac{h^j}{j!} \upsilon^{(j)}$, which satisfies

$$r_i^{\tau,h}(x) = r_{i-1}^{\tau,h}(x) + (L_i^h r_i^{\tau,h}(x) + F_i^{\tau,h}(x))\tau + \sum_{\rho=1}^{d_1} (M_{i-1}^{h\rho} r_{i-1}^{\tau,h}(x) + G_{i-1}^{\tau,h,\rho}(x))\xi_i^\rho \qquad \text{(rEq)}$$

  with $r_0^{\tau,h}(x) = 0$
- Where $F$ and $G$ are, respectively, certain combinations of differences between the $L^h$ and $\mathcal{L}^{(j)}$ and the $M^{h\rho}$ and $\mathcal{M}^{(j)\rho}$ operators acting on certain $\upsilon^{(0)}$, ..., $\upsilon^{(k)}$

# Proof of expansion result (Taylor expansion) III

- In particular, we know how $F$ and $G$ depend on $h$; then the estimate obtained for $(Eq_\tau^h)$ can be applied to $(rEq)$ with zero initial condition to obtain appropriate bound on error
- Final step is to take Sobolev embedding and then to show that the restriction to the grid of the continuous modifications of $w$, $v$, $\upsilon^{(j)}$ agree almost surely with the $\ell^2$-valued notion of solution to the space-time scheme □

# Summary

- We considered finite difference approximations on uniform grids in time and space for a second order linear SPDEs of parabolic type
- Under suitable regularity conditions, the spatial approximation can be accelerated to an arbitrarily high order using Richardson's method
- This relied on obtaining estimates in appropriate Sobolev spaces for the solutions to the schemes

# Final Remarks

- Using this program can: give estimates for accelerating the convergence of derivatives of solutions; consider symmetric differences (faster convergence)
- Results for degenerate parabolic equations are possible
- Future work: quantifying coefficients, truncation errors, other types of noise…

# References


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