Uncertainty quantification for mean field games in social interactions

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Outline

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$$\begin{cases} d\boldsymbol{X}(t) = b(\boldsymbol{X}(t), \alpha(t))dt + \sigma(\boldsymbol{X}(t))d\boldsymbol{W}_t \\ \boldsymbol{X}(t=0) = \boldsymbol{X}_0, \end{cases} \tag{1}$$

where

- the function b is the drift
- $\boldsymbol{W}_t$ is the standard Brownian motion
- α is the control

Remark

- If the control $\alpha = \alpha(t, \boldsymbol{X}_0)$; open-loop control
- If the control $\alpha = \alpha(t, \boldsymbol{X}(t))$; closed-loop control (feedback control)

Cost functional

$$J(\boldsymbol{X}_0, \alpha) = \mathbb{E}\left[\int_0^T f(\boldsymbol{X}, \alpha)dt + g(\boldsymbol{X}(T))\right],$$

- $f(t, \alpha(t), \boldsymbol{X}(t)) = \underbrace{\mathcal{U}(t, \alpha(t), \boldsymbol{X}(t))}_{\text{utility}} - \underbrace{\mathcal{C}(t, \alpha(t), \boldsymbol{X}(t))}_{\text{cost}}$
- g is differentiable and stands for the terminal reward.

Control Problem

$$(\mathcal{P}) \begin{cases} \sup\limits_{(\alpha(t))_t} J(\boldsymbol{X}_0, \alpha), \\ \text{subject to (1)} \end{cases} \quad (2)$$

Pontryagin's Maximum Principle

The Hamiltonian H is defined for all $t \in [0, T]$

$$H(t, \boldsymbol{p}, \boldsymbol{X}, \alpha) = \boldsymbol{p}^{tr} \cdot b(\boldsymbol{X}, \alpha) + \boldsymbol{\mu}^{tr} \cdot \sigma + f(\boldsymbol{X}, \alpha)$$

(1) Optimal control α^* gives $\boldsymbol{X}^*$ such that

$$H(t, \boldsymbol{p}^*, \boldsymbol{X}^*, \alpha^*) \geq H(t, \boldsymbol{p}^*, \boldsymbol{X}^*, \alpha) \quad \forall\, t \in [0, T]$$

(2) Co-state equation

$$d\boldsymbol{p}^{tr}(t) = -\nabla_{\boldsymbol{X}} H(t, \boldsymbol{p}, \boldsymbol{X}, \alpha)|_{(\boldsymbol{X}, \alpha) = (\boldsymbol{X}^*, \alpha^*)} + \boldsymbol{\mu}^{tr}(t) d\boldsymbol{W}_t$$

(3) Transversality condition $\boldsymbol{p}^{tr}(T) = \nabla_{\boldsymbol{X}} g(\boldsymbol{X}(T))$

Bellman Dynamic Programming

Let

$$a(t, \boldsymbol{x}) = \frac{1}{2}\mathrm{Tr}(\sigma(\boldsymbol{x}) \cdot \sigma^{tr}(\boldsymbol{x})) \tag{3}$$

The generator of diffusion is

$$L\varphi(t, \boldsymbol{x}, \boldsymbol{\alpha}) = a(t, \boldsymbol{x})\Delta\varphi + b(t, \boldsymbol{x}, \boldsymbol{\alpha}) \cdot \nabla\varphi \tag{4}$$

The value function $\boldsymbol{u} = \sup_{\boldsymbol{\alpha}\in\mathcal{A}} J(\boldsymbol{X}_0, \boldsymbol{\alpha})$ satisfies the HJB (Hamilton-Jacobi-Bellman equation):

$$\min_{\boldsymbol{\alpha}\in\mathcal{A}} \left\{ \frac{\partial \boldsymbol{u}}{\partial t} - L\boldsymbol{u}(t, \boldsymbol{x}, \boldsymbol{\alpha}) + f(t, \boldsymbol{x}, \boldsymbol{\alpha}) \right\} = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + a(t, \boldsymbol{x})\Delta\boldsymbol{u} + \underbrace{\max_{\boldsymbol{\alpha}\in\mathcal{A}} \{ b(t, \boldsymbol{x}, \boldsymbol{\alpha}) \cdot \nabla\boldsymbol{u} - f(t, \boldsymbol{x}, \boldsymbol{\alpha}) \}}_{H(t, \boldsymbol{x}, \nabla \boldsymbol{u})} = 0.$$

$\boldsymbol{X}$ modeled by a McKean-Vlasov process

$$\begin{cases} d\boldsymbol{X}(t) = b(\boldsymbol{X}(t), \boldsymbol{m}(t), \alpha(t))dt + \sigma(\boldsymbol{X}(t))d\boldsymbol{W}_t, \\ \boldsymbol{X}(0) = \boldsymbol{X}_0, \end{cases} \tag{5}$$

where

- $\boldsymbol{m}(t)$ is the the probability density of the state $\boldsymbol{X}$,
- $\alpha(t)$ is the feedback control
- $\boldsymbol{X}_0$ is independent to the Wiener process $(\boldsymbol{W}_t)_{t\geq 0}$.

Hypothesis

- Infinite number of players,
- Indistinguishability of the players,
- Regularity of the drift function $\boldsymbol{b}$.

$$A\varphi(\boldsymbol{x}) = -\sum_{i,j} \sigma_{ij}(\boldsymbol{x}) \frac{\partial^2 \varphi(\boldsymbol{x})}{\partial x_i \partial x_j}, \quad \text{its adjoint} \quad A^*\varphi(\boldsymbol{x}) = -\sum_{i,j} \frac{\partial^2 \left(\sigma_{ij}(\boldsymbol{x})\varphi(\boldsymbol{x})\right)}{\partial x_i \partial x_j}.$$

Cost functional

$$J(\boldsymbol{X}_0, \boldsymbol{m}, \alpha) = \mathbb{E}\left\{ \int_0^T f(\boldsymbol{X}(t), \boldsymbol{m}(t), \alpha(t))dt + g(\boldsymbol{X}(T), \boldsymbol{m}(T)) \right\}$$

Value function

and formulate the optimal control problem

$$\boldsymbol{u}(\boldsymbol{X}_0, \boldsymbol{m}(t)) = \inf_{\alpha(t)} J(\boldsymbol{X}_0, \boldsymbol{m}(t), \alpha(t)).$$

Problem

We wish to find the optimal control α and to characterize the value function $\boldsymbol{u}$ (the cost functional evaluated with the optimal trajectory α^*)

Let H be the Hamiltonian:

$$H(\boldsymbol{x}, \boldsymbol{m}, \boldsymbol{p}) = \inf_{\boldsymbol{\alpha}} \left(\boldsymbol{p}^{tr} \cdot b(\boldsymbol{X}(t), \boldsymbol{m}(t), \alpha(t)) + f(\boldsymbol{X}(t), \boldsymbol{m}(t), \alpha(t))\right)$$

Basic Setup

- Best response to the mean of the field :

$$\arg\min_{\boldsymbol{\alpha}} \left[H(\boldsymbol{x}, \boldsymbol{m}, \boldsymbol{p}) = \boldsymbol{p}^{tr} \cdot b(\boldsymbol{X}, \boldsymbol{m}, \alpha) + f(\boldsymbol{X}, \boldsymbol{m}, \alpha)\right]$$

- Dynamic equilibrium: No single-incentive
  Consistency of strategies (see Jovanovic' 82)

Pontryagin's Principle

Good points

- Gives control dynamics
- Covers nonlinear transversality condition

Limitations

- Hard to characterize the adjoint state dynamics
- Limited; $m = \mathbb{E}(\boldsymbol{X})$ works

Existence & Uniqueness

B. Djehiche et al. 2011.

Bellman Programming

Good points

- Works for general form of distribution m
- Theory is well developed

Limitations

- Time inconsistency when the transversality is nonlinear
- Only average behavior of all the agents

Existence & Uniqueness

Lasry & Lions 2006, Gueant 2009,

Huang-Caines-Malhamé 2007.

Numerical Implementation

Gomes et al 2009, Achdou et al. 2010, LaChapelle 2010.

Dynamic Programming for MFG

If

- the feedback control α does not affect the density m,
- the volatility function is linear on the state
- the initial distribution m_0 is normalized

Then, the stochastic optimization problem is called **Mean Field Games**

$$\begin{cases} -\dfrac{\partial u}{\partial t} + Au = H(x, m, \nabla u) & \text{in} \quad (0, T) \times \mathbb{R}^n, \\ \dfrac{\partial m}{\partial t} + A^* m + \text{div}\,(mH_p(x, m, \nabla u)) = 0 & \text{in} \quad (0, T) \times \mathbb{R}^n, \\ u(x, T) = g(x(T), m(T)), \quad m(x, 0) = m_0(x) & \text{in} \quad \mathbb{R}^n. \end{cases}$$

Some Remarks

- Approximate Nash equilibria may exist even when Nash equilibria do not! In this case one uses probabilistic techniques to prove that the MFG feedback gives the desired approximation.
- When the strategy of a single player affects the distribution, it is called Mean Field Type Control.

Outline

Optimal feedback effort I

Value function of a couple

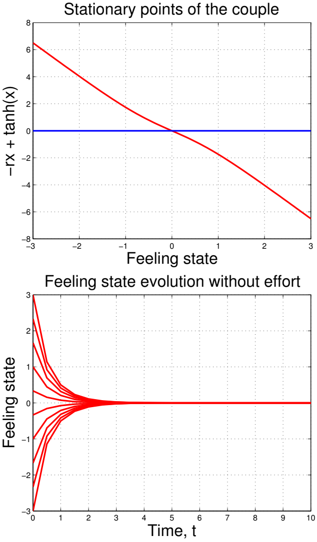
$$v(t,x) = \sup_{e(t,.)\geq 0} \mathbb{E}\left(g(x(T)) + \int_t^T [s(x(t')) - c(e(t'))]\, dt' \mid x(t) = x \right),$$

where

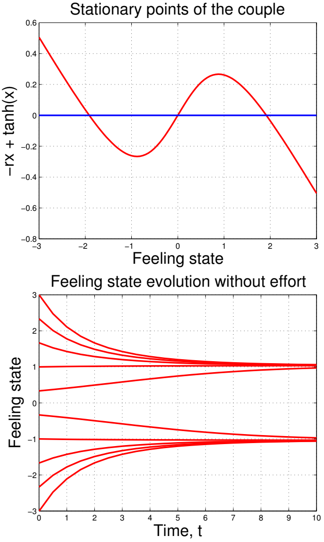
$$\begin{cases} dx = [-h(x) + ae]\, dt + \sigma dW(t), \quad \text{for} \quad t > 0 \\ x(t=0) = x_0 \quad \text{such that} \quad x_0 > \underline{x} + \varepsilon. \end{cases}$$

- $s(x)$ non-decreasing, concave and saturated at ∞
- $c(e)$ is $\mathcal{C}^2$ non-decreasing and strictly convex
- $a > 0$ the expected variation of the feeling in a short time
- $h(x) = rx - \tanh(x)$ (See Gottman et al. 2005)

Optimal feedback effort II



High type society



Low type society

Optimal feedback effort III

Dynamic programming: The Hamiltonian is

$$H(x,p) = -h(x)p + s(x) + \underbrace{\sup_e \{ap \cdot e - c(e)\}}_{\tilde{c}(ap)} .$$

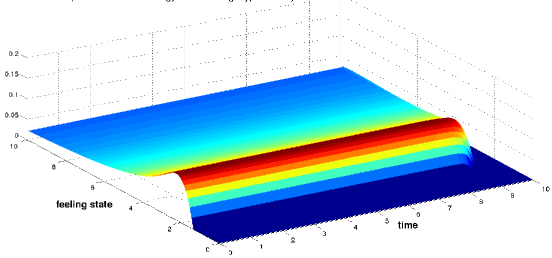
The optimal effort is

$$e^*(t,x) = \max\left(0\,,\ \tilde{c}'(av_x)\right) ,$$

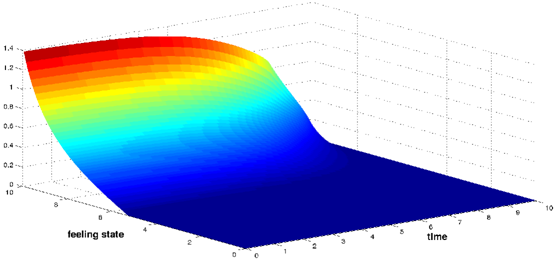
where

$$\begin{cases} v_t - h(x)v_x + s(x) + \tilde{c}(av_x) + \frac{\sigma^2}{2} v_{xx} = 0 \\ \text{with } \ v(T,x) = g(x), \end{cases}$$

Optimal feedback strategy effort in a high type society



Optimal feedback strategy for a couple in low type society



Mean Field

Feeling level $x(t)$ is modeled as a McKean-Vlasov Itô's SDE:

$$dx(t) = \left[-\hat{h}(x(t), m(t)) + a e(t)\right] dt + \sigma dW(t). \qquad (6)$$

$$(\mathcal{P}_2) \begin{cases} \sup\limits_{(e(t))_t} \; P_{\text{field}}(x_0, m, e), \\ \text{subject to the feeling state dynamics (6).} \end{cases}$$

where

$$P_{\text{field}}(x_0, m, e) = \mathbb{E}\left(g(x(T)) + \int_0^T \left[\hat{s}(x(t), m(t)) - c(e(t))\right] dt \right).$$

Mean-field equilibria

The mean-field equilibria are solution of the following backward-forward system for $(t, x) \in [0, T] \times \mathbb{R}$ is

$$\begin{cases} v_t - \hat{h}(x, m) v_x + \hat{s}(x, m) + \tilde{c}(a v_x) + \dfrac{\sigma^2}{2} v_{xx} = 0, \\ m_t + \partial_x \left[-m\hat{h} + am\tilde{c}'\left(a v_x\right)\right] - \dfrac{\sigma^2}{2} m_{xx} = 0, \\ m(t=0, x) = m_0(x) \text{ for } x \in \mathbb{R}, \\ v(t=T, x) = v_T(x) = g(x) \text{ for } x \in \mathbb{R}. \end{cases} \tag{7}$$

Result 1

The optimal level of effort for a long-term viability of the couple (during their lifetime) which keeps a happy relationship going is always greater than the effort level that would be chosen in a one-shot, i.e., $e^*(t,x) \ \geq \underline{e}$, where $c'(\underline{e}) = 0$.

Result 2

Even if x is below a certain threshold, Breaking Up is hard.

Result 3

If the mean-field has a tendency to the divorce states then, there is a distributive phenomenon for divorce. Breaking Up is not seen as a negative thing in that society because the majority is Doing it Too.

Outline

1. Mean Field Games
   - Stochastic Optimal control
   - Abstract Formulation of MFG
2. Mean Field Sentimental Games
3. Uncertainty Quantification

Wiener Choas Expansion

Wiener Chaos Space:

$$L^2(\mathbb{R},\mu) = \left\{ f : \int_{\mathbb{R}} f^2(s)\mu(ds) < \infty \right\}$$

where μ is the Gaussian measure

$$\mu(ds) = \rho(s)dx, \quad \rho(s) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{s^2}{2}).$$

- solution $X \equiv X(t, m, dW(t))$
- W(t) contains infinitely many independent Gaussian r.v.
- WCE decompose $X(t, m, dW(t))$ by orthogonal polynomials

Theorem (Cameron & Martin, 1947)

Assume that the state process $x(t)$ is in $L^2([0, T] \times \Omega, \mathbb{R})$ and is functional of the Brownian motion $\{W_s, \ \ 0 \leq s \leq t\}$, then

$$x(t, \omega) = \sum_{|\beta| < \infty} x_\beta(t) T_\beta(\xi(\omega)), \tag{8}$$

where T_β are random Wick polynomials.

Hermite Polynomials

$$H_p(y) = y^p + \sum_{k=0}^{p-1} \alpha_{p,k} y^k$$

Wiener Hermite Expansion

- $\{H_p\}_{p\geq 0}$ is a orthonormal basis of $L^2(0,T)$
- $\xi_p = \int_{\mathbb{R}} H_p(s)dW(s)$ are independent Gaussian r.v. with

$$\mathbb{E}[\xi_p] = 0 \quad \text{and} \quad Var(\xi_p) = ||H_p||^2_{L^2(0,T)}$$

- Let $\xi = (\xi_1,\ \xi_2\ ,\xi_3,\cdots)$ and $\beta = (\beta_1,\ \beta_2,\ \beta_3,\cdots)$ such that $|\beta| = \sum_{i\geq 1}\beta_i < \infty$

$$H_\beta(\xi) = \prod_{i=1}^{\infty} H_{\beta_i}(\xi_i)$$

then

$$x(t,dW) = \sum x_\beta(t)H_\beta(\xi) \quad x_\beta = \mathbb{E}[x(t,\xi)H_\beta(\xi)].$$

Moments:

$$\mathbb{E}[x](t,\xi) = x_0(t) \quad \mathbb{E}[x^2](t,\xi) = \sum_\beta |x_\beta|^2.$$

Wiener Hermite Expansion

Theorem

For each couple, given the type of the society, the best strategy:

$$\begin{cases} \sup_{(e_\beta(t))_t} \mathbb{E}\left(g(x(T)) + \int_0^T [\hat{s}(x(t), m^*(t)) - c(e_\beta(t))]\, dt \right), \\ \text{subject to} \\ \dot{x}_\beta(t) = \left[-\hat{h}(x(t), m(t)) + a e_\beta(t) \right] dt + \sigma \sum_{k=1}^{\infty} \sqrt{\beta_k} H_k(t) H_{\hat{\beta}_{(i)}}(t) \\ x_0(t) = x(0) \mathbb{1}_{\{\beta=0\}} \end{cases}$$

where

$$\hat{\beta}_{(i)}(j) = \begin{cases} \beta_j & \text{if} \quad i \neq j \\ \beta_j - 1 & \text{if} \quad i = j \end{cases}$$

Proof (Main Points)

- Multiply the dynamics

$$dx(t) = \left[-\hat{h}(x(t), m(t)) + ae(t)\right] dt + \sigma dW(t).$$

by H_β and take the expectation.

- Differentiate x_β using Mallavian derivative

$$dx_\beta = \sum_{k=1}^{\infty} \sqrt{\beta_k} H_k(s) H_{\hat{\boldsymbol{\beta}}_{(i)}} dW(s) \quad \text{for} \quad \leq s \leq t.$$

- Expand the Brownian motion in Fourier series

$$W(s) = \sum^{\infty} \xi_i \int_0^s H_i(\tau) d\tau.$$

References

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