Bayesian techniques for fatigue life prediction and for inference in linear time dependent PDEs

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Bayesian inference and model comparison for metallic fatigue data

Joint work (http://arxiv.org/abs/1512.01779) with

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Mechanical and structural components subjected to cyclic loading are susceptible to cumulative damage and eventual failure through an irreversible process called metal fatigue.

**Goal**
To predict the fatigue life of materials by providing a systematic approach to model calibration, model selection and model ranking with reference to stress-life (S-N) data drawn from a collection of records of fatigue experiments.
The 75S-T6 aluminum sheet specimens data set

Data are available on a collection of records of fatigue tests conducted at the Battelle Memorial Institute on 85 75S-T6 aluminum sheet specimens by means of a Krouse direct repeated-stress testing machine [1, table 3, pp.22–24]. The following data are recorded for each specimen:

- the maximum stress, $S_{\text{max}}$, measured in ksi units.
- the test ratio, $R$, defined as the minimum to maximum stress ratio.
- the fatigue life, $N$, defined as the number of load cycles at which fatigue failure occurred.
- a binary variable (0/1) to denote whether or not the test had been stopped prior to the occurrence of failure (run-out).
The 75S-T6 aluminum sheet specimens data set (cont.)
In 12 of the 85 experiments, the specimens remained unbroken when the tests were stopped.

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Max. Stress (ksi)</th>
<th>Load cycles</th>
<th>Test ratio</th>
<th>Type</th>
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<tbody>
<tr>
<td>B24M1</td>
<td>80.0</td>
<td>2,478,100</td>
<td>0.7</td>
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<td>B81M4</td>
<td>75.0</td>
<td>10,538,300</td>
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<td>0</td>
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<tr>
<td>B91M3</td>
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<td>B95M4</td>
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<td>71,700</td>
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<td>68,300</td>
<td>0.6</td>
<td>1</td>
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<tr>
<td>B93M1</td>
<td>80.5</td>
<td>99,000</td>
<td>0.6</td>
<td>1</td>
</tr>
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<td>B15M2</td>
<td>79.0</td>
<td>162,100</td>
<td>0.6</td>
<td>1</td>
</tr>
<tr>
<td>B23M4</td>
<td>79.0</td>
<td>181,600</td>
<td>0.6</td>
<td>1</td>
</tr>
<tr>
<td>B19M2</td>
<td>75.0</td>
<td>58,600</td>
<td>0.6</td>
<td>1</td>
</tr>
<tr>
<td>B19M3</td>
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<td>B16M1</td>
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<td>0</td>
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<tr>
<td>B19M1</td>
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<td>0</td>
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<tr>
<td>B35M3</td>
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<td>89,000</td>
<td>0.5</td>
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</table>
Statistical models for metallic fatigue data

In the Metallic Materials Properties Development and Standardization (MMPDS-01) Handbook, 2003, Battelle Memorial Institute, it is considered the so-called *logarithmic fit*

$$\mu(S_{eq}^{(lg)}) = A_1 + A_2 \log_{10}(S_{eq}^{(lg)} - A_3),$$

using the objective function,

$$e_{std} = \left( \frac{\sum_{i=1}^{n}(\log_{10}(n_i) - \mu(S_{eq}^{(lg)}))^2}{n - p} \right)^{1/2},$$

where $n$ is the number of data points and $p$ is the number of fitting parameters (namely $A_1$, $A_2$, $A_3$ and $q$).
The resulting estimated mean value function, without the run-outs, is given by

\[ \mu \left( S_{eq}^{(lg)} \right) = 7.71 - 2.17 \log_{10} \left( S_{eq}^{(lg)} - 31.53 \right), \]

where \( S_{eq}^{(lg)} = S_{max} \left( 1 - R \right)^{0.4633} \) and the value of the objective function is \( e_{std} = 0.3673 \).

The estimated fatigue limit is equal to \( 31.53/(2^{0.4633}) = 22.87 \) ksi, since the fatigue limit is the value of the maximum stress when the test ratio, \( R \), is equal to \(-1\) (the “fully reversed” condition).
Statistical models for metallic fatigue data (cont.)

Figure 1: Logarithmic fit of the 75S-T6 data set without run-outs.
We consider fatigue-limit models [2] and random fatigue-limit models [3] that are specially designed to allow the treatment of the run-outs (right-censored data).

Fatigue data obtained for particular test ratios need to be generalized to arbitrary test ratios. To this purpose the *equivalent stress* $S_{eq}$ is defined as $S_{eq}^{(q)} = S_{max} (1 - R)^q$, where $q$ is a fitting parameter.

We distinguish between the fatigue limit, which is a physical notion, and the fatigue limit parameter, which is an unknown parameter, expressed in the same scale as the equivalent stress and calibrated for different models.
Let $A_3$ be the *fatigue limit parameter*. At each equivalent stress with $S_{eq} > A_3$, the *fatigue life* $N$ is modeled with a lognormal distribution: $\log_{10}(N) \sim N(\mu(S_{eq}), \sigma(S_{eq}))$.

Assume that

- $\mu(S_{eq}) = A_1 + A_2 \log_{10}(S_{eq} - A_3)$, if $S_{eq} > A_3$
- $\sigma(S_{eq}) = \tau$.

Given the sample data, $n = (n_1, \ldots, n_m)$, we consider the *likelihood function* $L(A_1, A_2, A_3, \tau, q; n)$, given by

$$\prod_{i=1}^{m} \left[ \frac{g(\log_{10}(n_i); \mu(S_{eq}), \sigma(S_{eq}))}{n_i \log(10)} \right]^{\delta_i} \left[ 1 - \Phi \left( \frac{\log_{10}(n_i) - \mu(S_{eq})}{\sigma(S_{eq})} \right) \right]^{1-\delta_i},$$

where $\Phi$ is the cumulative distribution function of the standard normal distribution.
Model Ia (cont.)

where $g(t; \mu, \sigma) = \frac{1}{\sqrt{2\pi} \sigma} \exp \left\{ -\frac{(t-\mu)^2}{2\sigma^2} \right\}$, $\Phi$ is the cumulative distribution function of the standard normal distribution, and

$$\delta_i = \begin{cases} 
1 & \text{if } n_i \text{ is a failure} \\
0 & \text{if } n_i \text{ is a run-out} .
\end{cases}$$
Model Ia (cont.)

Figure 2: $A_1 = 7.38, A_2 = -2.01, A_3 = 35.04, q = 0.563, \tau = 0.527$. 

Bayesian techniques for fatigue life prediction and for inference in linear time dependent PDEs
Model Ib

We extend the Model Ia by allowing a non-constant standard deviation:

\[
\begin{align*}
\mu(S_{eq}) &= A_1 + A_2 \log_{10}(S_{eq} - A_3), \quad \text{if } S_{eq} > A_3 \\
\sigma(S_{eq}) &= 10(B_1 + B_2 \log_{10}(S_{eq})), \quad \text{if } S_{eq} > A_3.
\end{align*}
\]
Model Ib (cont.)

Figure 3:

\[ A_1 = 6.72, A_2 = -1.57, A_3 = 36.21, q = 0.551, B_1 = 4.55, B_2 = -2.89. \]
Profile likelihood

To assess the plausibility of a range of values of the fatigue limit parameter, $A_3$, we construct the profile likelihood [2, p.294]:

$$R(A_3) = \max_{\theta_0} \left[ \frac{L(\theta_0, A_3)}{L(\hat{\theta})} \right],$$

where $\theta_0$ denotes all parameters except for the fatigue limit parameter, $A_3$, and $\hat{\theta}$ is the ML estimate of $\theta$. 
Profile likelihood (cont.)

Figure 4: Profile likelihood estimates for the fatigue limit parameter, $A_3$, with Model Ia fit (blue curve) and Model Ib fit (red curve).
Model II

We extend the Model Ia to allow a random fatigue limit parameter as in [3]:

1. \( \mu(S_{eq}) = A_1 + A_2 \log_{10}(S_{eq} - A_3) \), if \( S_{eq} > A_3 \).
2. \( \sigma(S_{eq}) = \tau \).
3. The density of \( \log_{10}(A_3) \) is \( \phi(t; \mu_f, \sigma_f) \).
4. The conditional density of \( \log_{10}(N) \) given \( S_{eq} > A_3 \) is \( \phi(t; \mu(S_{eq}), \sigma(S_{eq})) \),

where \( \phi(t; \mu, \sigma) = \frac{1}{\sigma} \exp \left\{ \left( \frac{t-\mu}{\sigma} \right) - \exp \left( \frac{t-\mu}{\sigma} \right) \right\} \) is the smallest extreme value (sev) probability density function with location parameter \( \mu \) and scale parameter \( \sigma \) [4, Chapter 4].
Bayesian techniques for fatigue life prediction and for inference in linear time dependent PDEs

Metallic Fatigue Data

Statistical models

Model II (cont.)

Figure 5:

\[ A_1 = 6.51, \ A_2 = -1.47, \ \mu_f = 1.60, \ \sigma_f = 0.0385, \ q = 0.489, \ \tau = 0.085. \]
Bayesian approach – Model Ia

Model Ia: $A_1 \sim U(5, 13)$, $A_2 \sim U(-5, 0)$, $A_3 \sim U(24, 40)$, $q \sim U(0.1, 0.9)$, $\tau \sim U(0.1, 1.5)$.

Figure 6: Prior densities (red line) and approximate marginal posterior densities (blue line) for $A_1, A_2, q, \tau$ and $A_3$. 
Bayesian approach – Model Ia (cont.)

Figure 7: Contour plots of the approximate bivariate densities of each pair of parameters in Model Ia.
Bayesian techniques for fatigue life prediction and for inference in linear time dependent PDEs

Metallic Fatigue Data

Bayesian approach

Bayesian approach – Model Ib

Model Ib: $A_1 \sim U(4, 10)$, $A_2 \sim U(-4, 0)$, $A_3 \sim U(30, 40)$, $q \sim U(0.1, 0.9)$, $B_1 \sim U(2, 7)$, $B_2 \sim U(-5, -1)$.

Figure 8: Prior densities (red line) and approximate marginal posterior densities (blue line) for $A_1, A_2, q, B_1, B_2$ and $A_3$. 
Bayesian approach – Model Ib (cont.)

Figure 9: Contour plots of the approximate bivariate densities of each pair of parameters in Model Ib.
Bayesian approach – Model II

Model II: $A_1 \sim U(4, 10)$, $A_2 \sim U(-4, 0)$, $\mu_f \sim U(1.4, 1.8)$, $\sigma_f \sim U(0, 0.1)$, $q \sim U(0.1, 0.9)$, $\tau \sim U(0, 0.25)$.

Figure 10: Prior densities (red line) and approximate marginal posterior densities (blue line) for $A_1$, $A_2$, $q$, $\tau$, $\mu_f$ and $\sigma_f$. 
Bayesian approach – Model II (cont.)

Figure 11: Contour plots of the approximate bivariate densities of each pair of parameters in Model II.
Model comparison - classical information criteria

<table>
<thead>
<tr>
<th></th>
<th>Model Ia</th>
<th>Model Ib</th>
<th>Model II</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximum log-likelihood</td>
<td>-950.16</td>
<td>-920.51</td>
<td>-907.31</td>
</tr>
<tr>
<td>Akaike Information Criterion (AIC)</td>
<td>1910.3</td>
<td>1853.0</td>
<td>1826.6</td>
</tr>
<tr>
<td>Bayesian Information Criterion (BIC)</td>
<td>1922.5</td>
<td>1867.7</td>
<td>1841.3</td>
</tr>
<tr>
<td>AIC with correction</td>
<td>1911.1</td>
<td>1854.1</td>
<td>1827.7</td>
</tr>
</tbody>
</table>

\[
\text{AIC} = 2k - 2 \log L(\hat{\theta}|y) \\
\text{BIC} = k \log(n) - 2 \log L(\hat{\theta}|y) \\
\text{AIC}_c = \text{AIC} + \frac{2k(k+1)}{n-k-1}
\]
# Model comparison - Bayesian predictive criteria

<table>
<thead>
<tr>
<th></th>
<th>Model Ia</th>
<th>Model Ib</th>
<th>Model II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log marginal likelihood (Laplace)</td>
<td>-963.36</td>
<td>-940.25</td>
<td>-923.91</td>
</tr>
<tr>
<td>Log marginal likelihood (Laplace-Metropolis)</td>
<td>-963.51</td>
<td>-938.17</td>
<td>-923.76</td>
</tr>
<tr>
<td>Log pointwise predictive density (lppd)</td>
<td>-949.70</td>
<td>-920.52</td>
<td>-908.01</td>
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<tr>
<td>Deviance information criterion (DIC)</td>
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<td>1852.4</td>
<td>1827.1</td>
</tr>
<tr>
<td>Watanabe-Akaike information criterion (WAIC)</td>
<td>1912.1</td>
<td>1853.9</td>
<td>1826.4</td>
</tr>
<tr>
<td>5-fold cross-validation (elpd)</td>
<td>-955.70</td>
<td>-927.80</td>
<td>-914.07</td>
</tr>
</tbody>
</table>

\[
lppd = \sum_{i=1}^{n} \log \left( \frac{1}{S} \sum_{m=1}^{S} \rho(y_i | \theta^m) \right),
\]

where \( \theta^m, m = 1, \ldots, S, \) are draws from the posterior of \( \theta \).

\[
DIC = 2p_{DIC} - 2 \log p(y | \hat{\theta}_{Bayes})
\]

where \( p_{DIC} = 2 \left( \log p(y | \hat{\theta}_{Bayes}) - \frac{1}{S} \sum_{m=1}^{S} \log p(y | \theta^m) \right) \).

\[
WAIC = -2(lppd - p_{WAIC})
\]
Model comparison - Bayesian predictive criteria (cont.)

where

\[ p_{\text{WAIC}} = 2 \sum_{i=1}^{n} \left( \log \left( \frac{1}{S} \sum_{m=1}^{S} p(y_i|\theta^m) \right) - \frac{1}{S} \sum_{m=1}^{S} \log p(y_i|\theta^m) \right) \].

**K-fold cross-validation (elpd)**

Data are randomly partitioned into K disjoint subsets: \( \{y_k\}_{k=1}^{K} \).

Training set: \( \{y(\cdot k)\} = \{y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_K\} \).

For each training set compute the corresponding posterior distribution, \( p(\theta|y(\cdot k)) \).

The log predictive density for \( y_i \in y_k \) is computed using the training set \( \{y(\cdot k)\} \):

\[ \text{lpd}_i = \log \left( \frac{1}{S} \sum_{m=1}^{S} p(y_i|\theta^{k,m}) \right), \; i \in k, \]
where \( \{\theta_k^m\}_{m=1}^S \) are MCMC samples of the posterior \( p(\theta|y_{-k}) \).

Finally, compute the expected log predictive density (elpd):

\[
\text{elpd} = \sum_{i=1}^{n} \text{lpd}_i.
\]
Conclusions

- Models of various complexity that were designed to account for right-censored data are calibrated by means of the maximum likelihood method.
- Bayesian approach is considered and simulation-based posterior distributions are provided.
- Classical measures of fit based on information criteria and Bayesian model comparison were applied to determine which model might be preferred under different a priori scenarios.
- The classical approach and the Bayesian approach for model comparison have provided evidence in favor of Model II given the 75S-T6 data set.
Conclusions (cont.)

- An integrated set of computational tools has been developed for model calibration, cross-validation, consistency and model comparison, allowing the user to rank alternative statistical models based on objective criteria.
References


References (cont.)


A hierarchical Bayesian setting for an inverse problem in linear parabolic PDEs with noisy boundary conditions

Joint work (http://arxiv.org/abs/1501.04739) with

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We develop a hierarchical Bayesian setting to infer unknown parameters of linear parabolic partial differential equations, as an example of linear time dependent PDEs, under the assumption that noisy measurements are available in the interior of a domain of interest and for the unknown boundary conditions.

**Goal**
To solve the inverse problem based on the assumption that the boundary parameters are unknown and modeled by means of adequate probability distributions.
Bayesian techniques for fatigue life prediction and for inference in linear time dependent PDEs

Linear Parabolic PDEs with Noisy Boundary Conditions

Formulation of the problem

Consider the deterministic one-dimensional parabolic initial-boundary value problem:

\[
\begin{aligned}
\frac{\partial T}{\partial t} + L_{\theta} T &= 0, \quad x \in (x_L, x_R), \quad 0 < t \leq t_N < \infty \\
T(x_L, t) &= T_L(t), \quad t \in [0, t_N] \\
T(x_R, t) &= T_R(t), \quad t \in [0, t_N] \\
T(x, 0) &= g(x), \quad x \in (x_L, x_R)
\end{aligned}
\]  

(1)

where \( L_{\theta} \) is a linear second-order partial differential operator that takes the form

\[
L_{\theta} T = -\partial_x (a(x) \partial_x T) + b(x) \partial_x T + c(x) T,
\]
Formulation of the problem (cont.)

\( \theta(x) = (a(x), b(x), c(x))^{tr} \), and the partial differential operator \( \partial_t + L_\theta \) is parabolic.

Our main objective is to provide a Bayesian solution to an inverse problem for \( \theta \), where we assume that

i. \( \theta \) is unknown, while the initial condition \( g \) in the initial-boundary value problem is known;

ii. \( \theta \) is allowed to vary with the spatial variable \( x \).

Noisy readings of the function \( T(x, t) \) at the \( I + 1 \) spatial locations, including the boundaries, \( x_L = x_0, x_1, x_2, \ldots, x_{I-1}, x_I = x_R \), at each of the \( N \) times \( t_1, t_2, \ldots, t_N \), are assumed available.
Statistical setting

Let $Y_n := (Y_{0,n}, \ldots, Y_{I,n})^t$ denote the vector of observed readings at time $t_n$, and assume a statistical model with an additive Gaussian experimental noise $\epsilon_n$:

$$Y_n = \begin{bmatrix} T_L(t_n) \\ T(x_1, t_n) \\ \vdots \\ T(x_{I-1}, t_n) \\ T_R(t_n) \end{bmatrix} + \epsilon_n,$$  \hfill (2)

where $\epsilon_n \sim \mathcal{N}(0_{I+1}, \sigma^2 I_{I+1})$ for some measurement error variance $\sigma^2 > 0$. 

Bayesian techniques for fatigue life prediction and for inference in linear time dependent PDEs

Linear Parabolic PDEs with Noisy Boundary Conditions

Statistical setting
Statistical setting (cont.)

Let \( Y^I_n := (Y_{1,n}, \ldots, Y_{I-1,n})^t \) denote the vector of observed \((I-1)\times1\) data at the interior locations \(x_1, x_2, \ldots, x_{I-1}\) and let \( Y^B_n := (Y_{L,n}, Y_{R,n})^t \) be the vector of observed data at the boundary locations \(x_0, x_I\) at time \(t_n\).

Consider the time local problem, defined between consecutive measurement times, i.e.

\[
\begin{aligned}
\partial_t T + L_\theta T &= 0, \quad x \in (x_L, x_R), \quad t_{n-1} < t \leq t_n, \\
T(x_L, t) &= T_L(t), \quad t \in [t_{n-1}, t_n], \\
T(x_R, t) &= T_R(t), \quad t \in [t_{n-1}, t_n], \\
T(x, t_{n-1}) &= \hat{T}(x, t_{n-1}), \quad x \in (x_L, x_R),
\end{aligned}
\]

(3)
Statistical setting (cont.)

whose exact solution, denoted by $\hat{T}(\cdot, t_n)$, depends only on $\theta$, $\hat{T}(\cdot, t_{n-1})$ and the boundary values $\{T_L(t), T_R(t)\}_{t \in (t_{n-1}, t_n)}$.

**Lemma**

The probability density function (pdf) of $\mathbf{Y}_n^I$ is given by

$$
\rho(\mathbf{Y}_n^I|\theta, \hat{T}(\cdot, t_{n-1}), \{T_L(t), T_R(t)\}_{t \in (t_{n-1}, t_n)})
= \frac{1}{(\sqrt{2\pi}\sigma)^{I-1}} \exp\left(-\frac{1}{2\sigma^2} \|\mathbf{R}_{t_n}\|_{\ell^2}^2\right),
$$

where $\mathbf{R}_{t_n} := \begin{pmatrix} (\hat{T}(x_1, t_n) - Y_{1,n}, \ldots, \hat{T}(x_{I-1}, t_n) - Y_{I-1,n})^t \end{pmatrix}_{(I-1) \times 1}$

denotes the data residual vector at time $t = t_n$. 


Statistical setting (cont.)

Assume now that the Dirichlet boundary condition functions, $T_L(\cdot)$ and $T_R(\cdot)$, are well approximated by piecewise linear continuous functions in the time partition $\{t_n\}_{n=1,...,N}$.

In this way, only $2N$ parameters, say $T_L(t_n) = T_{L,n}$, $T_R(t_n) = T_{R,n}$, $n = 1, 2, \ldots, N$, suffice to determine the boundary conditions that are, in principle, infinite dimensional parameters.

Let $LBC_n$ denote the time nodes that determine the local boundary conditions $\{T_{L,n-1}, T_L, T_{R,n-1}, T_R\}$.
Lemma

The joint likelihood function of $\theta$ and the boundary parameters 
$\{LBC_n\}_{n=1,\ldots,N}$ is given by

$$
\rho(Y_1, \ldots, Y_N | \theta, \{LBC_n\}_{n=1,\ldots,N}) = \prod_{n=1}^{N} \frac{1}{(\sqrt{2\pi}\sigma)^{l-1}} \exp \left( -\frac{1}{2\sigma^2} \|R_t_n\|_{\ell^2}^2 \right) \times \frac{1}{2\pi\sigma^2} \exp \left( -\frac{1}{2\sigma^2} (T_{L,n} - Y_{L,n})^2 \right) \exp \left( -\frac{1}{2\sigma^2} (T_{R,n} - Y_{R,n})^2 \right).
$$

(5)

Notation: $T_L = (T_{L,1}, \ldots, T_{L,N})^{tr}$, $T_R = (T_{R,1}, \ldots, T_{R,N})^{tr}$, $Y_L = (Y_{L,1}, \ldots, Y_{L,N})^{tr}$ and $Y_R = (Y_{R,1}, \ldots, Y_{R,N})^{tr}$. 

Statistical setting (cont.)
The joint likelihood function (5) can be written as

\[
\rho(Y_1, \ldots, Y_N | \theta, T_L, T_R) = \left( \sqrt{2\pi\sigma} \right)^{-N(I+1)} \exp \left( -\frac{1}{2\sigma^2} \sum_{n=1}^{N} \| R_{t_n} \|_{\ell^2}^2 \right) \times \exp \left( -\frac{1}{2\sigma^2} \left[ \| T_L - Y_L \|_{\ell^2}^2 + \| T_R - Y_R \|_{\ell^2}^2 \right] \right). 
\]  

(6)
The marginal likelihood of $\theta$

Assume that the nuisance parameters $T_L$ and $T_R$ are independent Gaussian distributed:

$$
T_L \sim \mathcal{N}(\mu_L, \sigma^2_p I_N), \quad T_R \sim \mathcal{N}(\mu_R, \sigma^2_p I_N).
$$

(7)

Using (6), the marginal likelihood of $\theta$ is given by

$$
\rho(Y_1, \ldots, Y_N|\theta) = (\sqrt{2\pi}\sigma)^{-N(l+1)}(\sqrt{2\pi}\sigma_p)^{-2N} \int_{T_R} \int_{T_L} \exp \left(-\frac{1}{2\sigma^2} \sum_{n=1}^{N} \|R_{tn}\|_{\ell^2}^2 \right)
\times \exp \left(-\frac{1}{2\sigma^2}(T_L-Y_L)^{tr}(T_L-Y_L) - \frac{1}{2\sigma^2}(T_R-Y_R)^{tr}(T_R-Y_R) \right)
\times \exp \left(-\frac{1}{2\sigma^2_p}(T_L-\mu_L)^{tr}(T_L-\mu_L) - \frac{1}{2\sigma^2_p}(T_R-\mu_R)^{tr}(T_R-\mu_R) \right) dT_L dT_R,
$$

(8)

where $R_{tn}$ is approximated by

$$
\tilde{R}_{tn} = \left(B^n T_0 - Y_n^l \right) + A_{L,n}(\theta) T_L + A_{R,n}(\theta) T_R.
$$

(9)
Consider the heat equation (one–dim diffusion equation for \( T(x,t) \)):

\[
\begin{cases}
\partial_t T - \partial_x (\theta(x) \partial_x T) = 0, & x \in (x_L, x_R), \ 0 < t \leq t_N < \infty \\
T(0,t) = T_L(t), & t \in [0, t_N] \\
T(1,t) = T_R(t), & t \in [0, t_N] \\
T(x,0) = g(x), & x \in (x_L, x_R).
\end{cases}
\]  

(10)

**Goal**

To infer the thermal diffusivity, \( \theta(x) \), an unknown parameter that measures the rapidity of the heat propagation through a material, using a Bayesian approach, when the temperature is measured at \( l + 1 \) locations, \( x_0 = x_L, x_1, x_2, \ldots, x_{l-1}, x_l = x_R \), at each of the \( N \) times, \( t_1, t_2, \ldots, t_N \). Clearly, this problem is a special case of (1) where \( L_{\theta} = -\partial_x (\theta(x) \partial_x T) \) and \( \theta(x) > 0 \).
Consider a lognormal prior $\log \theta \sim \mathcal{N}(\nu, \tau)$, where $\nu \in \mathbb{R}$ and $\tau > 0$.

Assume noisy boundary measurements and a Gaussian distribution for the nuisance boundary parameters as in (7).

The non-normalized posterior density for $\theta$ is given by

$$
\rho_{\nu,\tau}(\theta|Y_1, \ldots, Y_N) \propto \frac{1}{\sqrt{2\pi \theta \tau}} \exp \left( - \frac{(\log \theta - \nu)^2}{2\tau^2} \right) \rho(Y_1, \ldots, Y_N|\theta),
$$

where $\rho(Y_1, \ldots, Y_N|\theta)$ is the marginal likelihood of $\theta$.

To assess the performance of our method we use a synthetic dataset, and assume that $\theta$ is a lognormal random variable with $\nu = \tau = 0.1$. 

Bayesian inference for thermal diffusivity

Figure 12: Example: Comparison between log-likelihoods (on the left) and log-posteriors (on the right) for $\theta$ using different numbers of observations, $N$, and different values of $\sigma$. 
Bayesian inference for thermal diffusivity (cont.)

Figure 13: Example: Comparison between log-likelihoods (on the left) and log-posteriors (on the right) for $\theta$ using different values of $\sigma$ and $\sigma_p$, with $N = 60$. 

Bayesian techniques for fatigue life prediction and for inference in linear time dependent PDEs
Bayesian inference for thermal diffusivity (cont.)

Figure 14: Example: Lognormal prior and approximated Gaussian posterior densities for $\theta$ where $\sigma_p = \sigma = 0.5$ and $N = 60$. 
Bayesian inference for thermal diffusivity (cont.)

Introduce three experimental setups (es’s) of interest:

es1) $\xi$ consists of three non-overlapping time intervals, with the same length, which cover the entire observational period $[0, 1]$;

es2) $\xi$ consists of the five inner thermocouples;

es3) $\xi$ is the combination of the two previous experimental setups, es1 and es2.
Bayesian inference for thermal diffusivity (cont.)

Figure 15: Example: The expected information gain compared with the information divergence for the synthetic dataset, for the three time intervals experimental setup (es1).
Bayesian inference for thermal diffusivity (cont.)

Figure 16: Example: The expected information gain compared with the information divergence for the synthetic dataset, for the five inner thermocouples experimental setup (es2).
Bayesian inference for thermal diffusivity (cont.)

![Graph](image)

**Figure 17**: Example: The expected information gain computed for the combination (es3) of the three time intervals and three out of five inner thermocouples experimental setups.
Bayesian techniques for fatigue life prediction and for inference in linear time dependent PDEs

- Linear Parabolic PDEs with Noisy Boundary Conditions
- Example - inference for thermal diffusivity

Many thanks for your attention!
Table 1: Maximum posterior estimates for Model Ia.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$q$</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laplace</td>
<td>7.38</td>
<td>-2.01</td>
<td>35.04</td>
<td>0.563</td>
<td>0.527</td>
</tr>
<tr>
<td>Laplace-Metropolis</td>
<td>7.39</td>
<td>-2.02</td>
<td>35.07</td>
<td>0.563</td>
<td>0.517</td>
</tr>
</tbody>
</table>

Table 2: MCMC posterior empirical mean estimates with their standard deviations.

<table>
<thead>
<tr>
<th></th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$q$</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>7.66</td>
<td>-2.18</td>
<td>34.31</td>
<td>0.557</td>
<td>0.549</td>
</tr>
<tr>
<td>SD</td>
<td>0.51</td>
<td>0.35</td>
<td>1.18</td>
<td>0.021</td>
<td>0.049</td>
</tr>
</tbody>
</table>
Bayesian techniques for fatigue life prediction and for inference in linear time dependent PDEs

Table 3: Maximum posterior estimates for Model Ib.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$q$</th>
<th>$B_1$</th>
<th>$B_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laplace</td>
<td>6.72</td>
<td>-1.57</td>
<td>36.21</td>
<td>0.551</td>
<td>4.55</td>
<td>-2.89</td>
</tr>
<tr>
<td>Laplace-Metropolis</td>
<td>6.69</td>
<td>-1.55</td>
<td>36.19</td>
<td>0.551</td>
<td>4.76</td>
<td>-3.00</td>
</tr>
</tbody>
</table>

Table 4: MCMC posterior empirical mean estimates with their standard deviations.

<table>
<thead>
<tr>
<th></th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$q$</th>
<th>$B_1$</th>
<th>$B_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>6.90</td>
<td>-1.67</td>
<td>35.50</td>
<td>0.541</td>
<td>4.46</td>
<td>-2.83</td>
</tr>
<tr>
<td>SD</td>
<td>0.26</td>
<td>0.16</td>
<td>0.69</td>
<td>0.024</td>
<td>0.55</td>
<td>0.32</td>
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</tbody>
</table>
**Table 5**: Maximum posterior estimates for Model IIb.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$\mu_f$</th>
<th>$\sigma_f$</th>
<th>$q$</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laplace</td>
<td>6.51</td>
<td>-1.48</td>
<td>1.60</td>
<td>0.0386</td>
<td>0.489</td>
<td>0.0853</td>
</tr>
<tr>
<td>Laplace-Metropolis</td>
<td>6.57</td>
<td>-1.51</td>
<td>1.60</td>
<td>0.0398</td>
<td>0.490</td>
<td>0.0856</td>
</tr>
</tbody>
</table>

**Table 6**: MCMC posterior empirical mean estimates with their standard deviations.

<table>
<thead>
<tr>
<th></th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$\mu_f$</th>
<th>$\sigma_f$</th>
<th>$q$</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>6.60</td>
<td>-1.52</td>
<td>1.60</td>
<td>0.0428</td>
<td>0.489</td>
<td>0.0901</td>
</tr>
<tr>
<td>SD</td>
<td>0.21</td>
<td>0.13</td>
<td>0.012</td>
<td>0.008</td>
<td>0.019</td>
<td>0.025</td>
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</tbody>
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