A potpourri of results from the KAUST SRI-UQ

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January 8, SRI-UQ Workshop, KAUST, KSA
1. Monte Carlo (MC) and Multilevel Monte Carlo (MLMC)
2. An Adaptive MLMC algorithm for Itô SDEs
3. Hybrid MLMC for Pure Jump Processes
4. Continuation MLMC and Optimized hierarchies
5. Multi Index Monte Carlo
Motivational Example: Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\mathcal{D} \subset \mathbb{R}^d$ be a bounded convex polygonal domain. The solution $u : \mathcal{D} \times \Omega \to \mathbb{R}$ here solves almost surely (a.s.) the following equation:

$$-\nabla \cdot (a(x; \omega) \nabla u(x; \omega)) = f(x; \omega) \quad \text{for } x \in \mathcal{D},$$

$$u(x; \omega) = 0 \quad \text{for } x \in \partial \mathcal{D}. \quad (1)$$

Goal: to approximate $\mathbb{E}[g] \in \mathbb{R}$ where $g = \Psi(u)$ for some sufficiently “smooth” $u$ (solution of a random PDE/stochastic differential equation) and a given functional $g$.

Assumption: We assume we have an approximation of $u$, say $\bar{u}_h$ (time discretization, FEM, FD, FV, . . . ) based on discretization parameter $h$.

Notation: $g_h$ is the approximation of $g$ calculated using a discretization defined by $h$. 
Monte Carlo (MC) approximates expectations by sample averages of i.i.d. approximate realizations

\[ E[g] \approx E[g_h] \approx \frac{1}{M} \sum_{m=1}^{M} g_h(\omega_m). \]

**Error splitting:**

\[ E[g] - \frac{1}{M} \sum_{m=1}^{M} g_h(\omega_m) = \mathcal{E}_{Bias}(h) + \mathcal{E}_{Stat}(M) \]

\[ |\mathcal{E}_{Bias}(h)| = |E[g] - E[g_h]| \leq Ch^w \]

\[ |\mathcal{E}_{Stat}(M)| = |E[g_h] - \frac{1}{M} \sum_{m=1}^{M} g_h(\omega_m)| \lesssim c_0 \sqrt{\frac{\text{Var}[g_h]}{M}} \]

The last estimate is motivated in probability by a Central Limit Theorem.
Monte Carlo complexity analysis

Let us assume now that the computational work to solve for each sample of $g_h$ is $O(h^{-d\gamma})$. Thus, we have the following estimates

**Total work:** $W \lesssim M h^{-d\gamma}$

**Total error:** $|\mathcal{E}_{\text{Bias}}(h)| + |\mathcal{E}_{\text{Stat}}(M)| \leq C_1 h^w + \frac{C_2}{\sqrt{M}}$

We want now to choose optimally $h$ and $M$. We thus minimize the computational work subject to an accuracy constraint, i.e. we solve

$$\begin{cases} 
\min_{h,M} & M h^{-d\gamma} \\
\text{s.t.} & C_1 h^w + \frac{C_2}{\sqrt{M}} \leq TOL 
\end{cases}$$

The resulting complexity (error versus computational work) is then

$$W \lesssim TOL^{-(2+d\gamma/w)}$$
**Seminal works:** [Giles06, Heinrich01].

**Construction:** Take $\beta > 1$ and for each $\ell = 1, 2, \ldots$ use discretizations with $h_{\ell} = h_0 \beta^{-\ell}$.

Recall the standard MLMC difference operator

$$\Delta g_{\ell} = \begin{cases} 
g_{h_0} & \text{if } \ell = 0, \\
g_{h_\ell} - g_{h_{\ell-1}} & \text{if } \ell > 0. 
\end{cases}$$

Observe the telescopic identity

$$E[g] \approx E[g_{h_L}] = \sum_{\ell=0}^{L} E[\Delta g_{\ell}].$$
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E[g] \approx E[g_{h_L}] = \sum_{\ell=0}^{L} E[\Delta g_\ell].
\]

Then, using MC to approximate each level independently, the MLMC estimator can be written as

\[
A_{\text{MLMC}} = \sum_{\ell=0}^{L} \frac{1}{M_\ell} \sum_{m=1}^{M_\ell} \Delta g_\ell(\omega_\ell, m).
\]
Recall: With Monte Carlo we have to satisfy

\[ \text{Var}[A_{MC}] = \frac{1}{M_L} \text{Var}[g_L] \approx \frac{1}{M_L} \text{Var}[g] \leq TOL^2. \]

Main point: MLMC reduces the variance of the deepest level using samples on coarser (less expensive) levels!

\[ \text{Var}[A_{MLMC}] = \frac{1}{M_0} \text{Var}[g_0] \]

\[ + \sum_{\ell=1}^{L} \frac{1}{M_\ell} \text{Var}[\Delta g_\ell] \leq TOL^2. \]  

Observe: Level 0 in MLMC is usually determined by both stability and accuracy, i.e.

\[ \text{Var}[\Delta g_1] \ll \text{Var}[g_0] \approx \text{Var}[g] < \infty. \]
For every level $\ell$, assume:

**Assumption 1 (Bias):**

$$|E[g - g_\ell]| \leq C\beta^{-w\ell},$$

**Assumption 2 (Variance):**

$$\text{Var}[\Delta g_\ell] \leq C\beta^{-s\ell},$$

**Assumption 3 (Work):**

$$\text{Work}(\Delta g_\ell) \leq C\beta^{d\gamma\ell},$$

for positive constants $C$, $\gamma$, $w$ and $s < 2w$. 

Example:

For the smooth linear elliptic PDE (1) approximated with multilinear piecewise continuous FEM we have:

$$2w = s = 4 \text{ and } 1 \leq \gamma \leq 3.$$
Assumptions for MLMC

For every level \( \ell \), assume:

**Assumption \( \tilde{1} \) (Bias):**

\[ |E[g - g_\ell]| \leq C \beta^{-w \ell}, \]

**Assumption \( \tilde{2} \) (Variance):**

\[ \text{Var} [\Delta g_\ell] \leq C \beta^{-s \ell}, \]

**Assumption \( \tilde{3} \) (Work):**

\[ \text{Work}(\Delta g_\ell) \leq C \beta^{d \gamma \ell}, \]

for positive constants \( C, \gamma, w \) and \( s < 2w \).

**Example:** For the smooth linear elliptic PDE (1) approximated with multilinear piecewise continuous FEM we have:

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$$\text{Work}(\Delta g_\ell) \leq C \beta^d \gamma_\ell,$$

for positive constants $C, \gamma, w$ and $s < 2w$.

**Example:** For the smooth linear elliptic PDE (1) approximated with multilinear piecewise continuous FEM we have:

$$2w = s = 4 \text{ and } 1 \leq \gamma \leq 3.$$  

**Total Work MLMC:**

$$\text{Work(MLMC)} = \sum_{\ell=0}^{L} M_\ell \text{ Work}(\Delta g_\ell)$$

$$\propto TOL^{-2} \left( \sum_{\ell=0}^{L} \sqrt{\text{Work}(\Delta g_\ell) \text{ Var}[\Delta g_\ell]} \right)^2$$
We choose the number of levels to bound the bias

\[ |E[g - g_L]| \propto \beta^{-Lw} \leq CTOL \quad \Rightarrow \quad L \geq \frac{\log(TOL^{-1}) - \log(C)}{w \log(\beta)}, \]

and then the samples \((M_\ell)_L^{\ell=0}\) to minimize the total work s.t. (2) [Giles et al., 08,11]:

\[
\text{Work(MLMC)} = \begin{cases} 
O(TOL^{-2}), & s > d\gamma, \\
O(TOL^{-2} (\log(TOL^{-1}))^2), & s = d\gamma, \\
O(TOL^{-(2 + \frac{d\gamma - s}{w})}), & s < d\gamma. 
\end{cases}
\]

Recall: \(\text{Work(MC)} = O(TOL^{-(2 + \frac{d\gamma}{w})}).\)

Today’s questions: How to extend MLMC into non-uniform, adaptive discretization settings? Is it worth it?
References:


For the Ito SDE

\[ dX_t = a(X_t, t) \, dt + \sum_{k=1}^{K} b^k(X_t, t) \, dW_t^k, \quad 0 < t < T, \]

\[ X_0 = x_0, \quad (3) \]

and \( g : \mathbb{R}^d \rightarrow \mathbb{R} \), approximate \( E[g(X_T)] \) to a given accuracy \( TOL \) and with prescribed confidence. \( W_t \) is a \( K \)-dimensional Wiener process.

**Applications:** Diffusion processes, Langevin Dynamics, Computational Finance, Crowd Flows,…
1. Forward Euler scheme on a grid $t_0 = 0 < t_1 < \ldots < t_N = T$

$$\bar{X}_{n+1} = \bar{X}_n + a(\bar{X}_n, t_n)\Delta t_n + \sum_{k=1}^{K} b^k(\bar{X}_n, t_n)\Delta W_n^k$$

This gives approximate realisations $\bar{X}_T(\omega)$.

Here $\Delta t_n = t_{n+1} - t_n$ and independent Wiener increments

$$\Delta W_n^k = W_{n+1}^k - W_n^k \sim N(0, \Delta t_n) \sim \sqrt{\Delta t_n}N(0, 1).$$

2. Monte Carlo estimator:

$$E[g(X_T)] \approx \sum_{i=1}^{M} \frac{g(\bar{X}_T(\omega_i; \Delta t))}{M}$$
Given $TOL_T$, use adaptive refinements to generate stochastic grids $t_0 = 0 < t_1(\omega) < \ldots < t_N = T$ to create realizations $\overline{X}_T(\omega; \Delta t(\omega))$.

Why? Non-smooth $a(X_s, s)$ or $b(X_s, s)$ can decrease convergence rates.

How? Use a posteriori SDE weak error density [STZ01], [MSTZ06],[MSTZ08].

For $\max \Delta t(TOL) \to 0$ as $TOL \to 0$ we have

$$E \left[ g(X_T) - g(\overline{X}_T) \right] \simeq \int_0^T E[\Delta t(s)\rho_W(\overline{X}_s, s)]ds$$

One also has a strong error density [HHT14]

$$E \left[ (g(X_T) - g(\overline{X}_T))^2 \right] \simeq \int_0^T E[\Delta t(s)\rho_S(\overline{X}_s, s)]ds$$
Adaptive refinements start from a coarse initial grid, and

1. compute solution and error indicators \( r_n \) for each time step \( n \),

2. as long as

\[
\max_n r_n \geq C_S \frac{TOL_T}{E[N]}, \tag{4}
\]

3. refine all time steps s.t.

\[
r_n \geq C_R \frac{TOL_T}{E[N]}, \tag{5}
\]

refine sampling of \( W \) by Brownian bridges, and go to (1).

\[
r_n = \rho_n \Delta t_n^2
\]
How to apply the multilevel Monte Carlo idea with adaptive time stepping?

Let the tolerance in the adaptive algorithm define the hierarchy!

Grid hierarchy defined by

\[ TOL_{T,\ell} = \frac{TOL_{T,0}}{2^\ell}. \]

When is adaptivity useful for multilevel Monte Carlo?

- Non-smooth \( a(X_s, s) \) or \( b(X_s, s) \) can decrease both weak and strong convergence rates for uniform grids, but also affects proportionality constants.
- Jentzen & Kloeden pointed out that stability problems on coarse grids can ruin multilevel Monte Carlo convergence for stiff SDEs.
Developed the first adaptive MLMC algorithm. Theoretical analysis of our adaptive MLMC algorithm proves

- **Stopping**: The adaptive MLMC Algorithm stops.
- **Asymptotic Normality**: Show CLT for MLMC based on Lindeberg-Feller’s CLT theorem.
- **Asymptotic Accuracy**: With prescribed confidence, the Error is bounded by TOL.
- **Complexity**: Essentially the same as MLMC has with uniform time steps in smooth problems.

All these results hold for our **adaptive** ($\Delta t(t, \omega)$ non uniform), **non-adapted** ($\bar{X}$ depends on the future values of $\mathcal{W}$ through $\Delta t$) discretizations.
Drift Singularity, gains from adaptive MLMC

Adaptive realization $p=0.5$

$X^{(2)}_t(\omega), X^{(6)}_t(\omega), X^{(4)}_t(\omega), X^{(8)}_t(\omega)$

Adaptive realization $p=0.67$

Adaptive realization $p=0.75$

$dX_t = \frac{X_t}{5|t - \alpha|^p} dt + 0.5 X_t dW_t, X_0 = 1, \alpha \sim U(1/4, 3/4)$
Conclusions

- Extended single level adaptive time stepping algorithms to the MLMC setting.
- Showed a CLT result that allows prescription of confidence level in our computations.
- Asymptotic estimates describe the behavior of the resulting adaptive algorithms, numerical experiments confirm the predicted bounds.
- Extension to stopped [DMSST05] and jump diffusions [MSTZ08] is direct.
Figure: Semilogarithmic plot showing 20 exact i.i.d. paths of 3 biochemical species in a problem from genomics.
Motivational Example: Gene transcription and translation

In [Anderson2012] the following example is proposed:

- $G \xrightarrow{25} G + M$, a single gene is being transcribed into mRNA.
- $M \xrightarrow{1000} M + P$, mRNA is then being translated into proteins.
- $P + P \xrightarrow{0.001} D$, finally the proteins produce stable Dimers.
- $M \xrightarrow{0.1} \emptyset$, $P \xrightarrow{1} \emptyset$ degradation of mRNA and proteins, respectively.

Initial state: $X(0) = (0, 0, 0)$, where $X_1, X_2, X_3$ give the molecular counts of the mRNA, proteins, and dimers, respectively.

**Goal:** We want to estimate the expected number of Dimers at time $T = 1$ up to certain tolerance, with high probability.
Mathematical statement of the problem

Let $X$ be a Pure Jump Process $X = (X_1, \ldots, X_d) : [0, T] \times \Omega \rightarrow \mathbb{Z}^d_+$ described by

- Finite number of possible reaction channels $\nu_j \in \mathbb{Z}^d$.

$$x \in \mathbb{Z}^d_+, \ x \rightarrow x + \nu_j$$

- and its corresponding propensity functions, $a_j : \mathbb{R}^d \rightarrow \mathbb{R}^+$ s.t.

$$P \left( X(t + dt) = x + \nu_j \mid X(t) = x \right) = a_j(x)dt + o(dt),$$

Motivation: accurately approximate the expected value

$$E[g(X(T))],$$

for some given observable $g : \mathbb{R}^d \rightarrow \mathbb{R}$.

Applications: Systems biology, complex reaction networks, biochemical kinetics, stochastic epidemic spread modeling, . . . .
References: MLMC for Pure Jump Processes


See Alvaro, Pedro & Chiheb’s posters for further details.
• **Exact algorithms** like the Stochastic Simulation Algorithm (SSA) [Gillespie76] and the Modified Next Reaction Method (MNRM) [Anderson07] sometimes are too expensive for path simulation. Just consider the inter-arrival time between transitions $\tau_{\text{SSA}}|X(t) = x \sim \text{exponential}(\sum_j a_j(x))$.

• **Approximate algorithms** that evolve with fixed time steps, like the Tau-leap, may be faster [Gillespie01]. Two drawbacks: i) time discretization errors ii) may lead to negative population numbers, i.e., non-physical results. Pre-leap: adjust adaptively the time step to control the one-step exit probability.

[Moraes, Vilanova, T.14]:
– single level hybrid algorithm that, at each time step, adaptively switches between the SSA and the Tau-leap to min comp. cost.
– related hybrid adaptive MLMC algorithms for error control
– a variance reduction technique based on Kurtz representation and a deterministic time change.
Figure: Actual work for each one of the one hundred adaptive runs. Our hybrid MLMC is 10 times faster than the SSA.
**Figure:** TOL versus the actual computational error. The numbers above the straight line show the percentage of runs that had errors larger than the required tolerance. We observe that in all cases the computational error follows the imposed tolerance closely with the expected confidence of 95%.
**Figure:** Speed-ups of order $10^2 - 10^4$ are obtained in stiff problems using our novel reaction-splitting Multilevel Monte Carlo method. We developed also a novel Control Variate based on Kurtz representation and a deterministic-time change approximation.

\[ \emptyset \xrightarrow{k} X_n, \quad X_n \xrightarrow{c} Y, \quad n = 1, \ldots, N = 200 \]

\[ Y \xrightarrow{a} \emptyset, \quad Y \xrightarrow{b} 50 \, Y \]
Figure: Speed-ups of order $10^2 - 10^4$ are obtained in stiff problems using our novel reaction-splitting Multilevel Monte Carlo method. We developed also a novel Control Variate based on Kurtz representation and a deterministic-time change approximation.

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Figure: Speed-ups of order $10^3$ are obtained in stiff problems using our novel reaction-splitting Multilevel Monte Carlo method. We developed also a novel Control Variate based on a deterministic-time change approximation. Example adapted from [Cao-Gillespie-Petzold,2005].

$$X_1 \xrightarrow{c_1/c_2} X_2 \xrightarrow{c_3} X_3 \xrightarrow{c_4} \emptyset, \quad c_2 \gg c_3 > c_4.$$
The computational complexity of this method is of order $O(TOL^{-2})$, and therefore, it can be seen as a variance reduction of the SSA method, which has the same complexity. This represents an important advantage of the hybrid tau-leap with respect to the pure tau-leap in the MLMC context.

Our algorithm provides the elements for the simulation setting (i.e., initial time mesh, number of levels, one-step exit probabilities and number of coupled hybrid paths at each level) that optimizes the computational work.

For reaching this optimality, we derived novel formulas based on dual-weighted residual to estimate the variance of the difference of the observables between two consecutive levels in coupled hybrid paths.
Goal: compute $E[g]$ where $g = g(u)$. Here $g$ is either a bounded linear functional or a Lipschitz functional with respect to $u$, and $u$ solves a stochastic equation.

Example:

$$- \nabla \cdot (a(x; \omega) \nabla u(x; \omega)) = f(x; \omega) \quad \text{for } x \in \mathcal{D} := [0, 1]^d,$$

$$u(x; \omega) = 0 \quad \text{for } x \in \partial \mathcal{D},$$

and

$$g(u) = \int_{\mathcal{D}} k(x) u(x) dx,$$

for sufficiently smooth $a, f, k$.

References:


Following the standard MLMC approach, we introduce a hierarchy of $L + 1$ meshes defined by decreasing mesh sizes $\{h_\ell\}_{\ell=0}^{L}$ and we denote the approximation of $g$ using mesh size $h_\ell$ by $g_\ell$. We then write the MLMC estimator as

$$A = \frac{1}{M_0} \sum_{m=1}^{M_0} g_0(\omega_0,m) + \sum_{\ell=1}^{L} \frac{1}{M_\ell} \sum_{m=1}^{M_\ell} (g_\ell(\omega_\ell,m) - g_{\ell-1}(\omega_\ell,m)).$$

(7)

We assume positive constants $Q_W, Q_S, q_1, q_2, d$ and $\gamma$ s.t.

$$|\mathbb{E}[g_\ell - g]| \approx Q_W h_\ell^{q_1},$$  

(8a)

$$\text{Var}[g_\ell - g_{\ell-1}] := V_\ell \approx Q_S h_{\ell-1}^{q_2},$$  

(8b)

$$\text{Work per sample of level } \ell := W_\ell \approx h_\ell^{-d\gamma}.$$  

(8c)

**Goal:** Choose $\{h_\ell\}_{\ell=0}^{L}, \{M_\ell\}_{\ell=0}^{L}$ optimally to minimize work while meeting prescribed accuracy and confidence constraints.
Examples

Examples for $q_1, q_2$:

- $q_1 = q_2 = 1$ for an SDE with Euler-Maruyama approximation.
- In our example: a PDE with smooth random coefficients and for piecewise linear or piecewise bilinear continuous finite element approximations we have $q_1 = 2$ and $q_2 = 4$.

Examples for $\gamma$:

- $\gamma = 1$ for an SDE with Euler-Maruyama approximation.
- In our PDE example: $d = 3$ and $\gamma = 3$ for a naive Gaussian elimination implementation. Moreover, Using an Iterative solver has $\gamma \approx 1$ while using Direct solver has $\gamma \approx 1.5$.

We define: $\chi = \frac{q_2}{d \gamma}$ and $\eta = \frac{q_1}{d \gamma}$.

In our PDE example: $\chi \approx 1.34$ for iterative solver and $\chi \approx 0.89$ for direct solver.
Conclusions

- Showed asymptotic normality of MLMC estimator under certain conditions through the use of Lindeberg central limit theorem. We use this in the formulation of our MLMC algorithm and the work optimization problem.

- Computational saving through better tolerance splitting between bias and statistical error contributions.

- A more stable continuation MLMC algorithm with a small overhead. In CMLMC, reusing samples does not introduce significant computational savings.

- We show that geometric hierarchies are near-optimal. Moreover, we derive the computational complexity with known rates and constants.

See Erik von Schwerin’s talk/posters for details
Variance reduction: Further potential
Multi Index Monte Carlo (MIMC) and Multi Index Stochastic Collocation (MISC)

References:


(See Fabio’s & Abdul-Lateef talks/posters )
We want to compute $E[S] \in \mathbb{R}$. Assume $S_\alpha$ is a discretization of $S$ with discretization parameters of the form

$$h_i = \beta_i^{-\alpha_i}$$

for $i \in 1, 2, \ldots, d$. Assume $E[S_\alpha] \to E[S]$ as $\min_{1 \leq i \leq d} \alpha_i \to \infty$. Define

$$\Delta_i S_\alpha = \begin{cases} S_\alpha & \text{if } \alpha_i = 0, \\ S_\alpha - S_{\alpha-e_i} & \text{if } \alpha_i > 0, \end{cases} \quad (9)$$

and let $\Delta S_\alpha = (\otimes_{i=1}^d \Delta_i) S_\alpha$. Then the MIMC estimator can be written as

$$A_{\text{MIMC}} = \sum_{\alpha \in \mathcal{I}} \frac{1}{M_\alpha} \sum_{m=1}^{M_\alpha} \Delta S_\alpha(\omega_\alpha, m),$$

with index set $\mathcal{I} \subset \mathbb{N}^d$ and independent samples.
Recall that the work complexity of MC is $O\left(TOL^{-5}\right)$.
Numerical test: Running time, 4D PDE problem

A Similar d=4 problem.
MIMC may perform better than MLMC. Especially in higher dimensions.

MIMC requires mixed regularity between discretization parameters.

Just like MLMC reduces to MC when $L = 0$. MIMC reduces to MLMC when $d = 1$ or we do not have mixed regularity.

MIMC can take advantage of non-isotropic behavior in different directions.

A different set of regularity assumptions would yield a different optimal index set $\mathcal{I}$.

A direction does not have to be a spatial dimension. It can represent any form of discretization parameter. Example: Number of particles See Abdul-Lateef’s talk/poster
Heat exchange processes are required in next to all industries, including power generation.

In most cooling water systems the materials exposed to the water suffer from fouling, which is the formation of a film that covers the surfaces in contact with the water.

In general, fouling causes important operation and maintenance costs, including delays in startups.

Figure: Left: Fouling in heat exchangers. Right: Fouling deposition in cooling water systems of oil refineries.
Motivation

- Fouling accumulation has a marked stochastic nature and also the available measurements are indirect and noisy.
- Modeling of fouling deposition has great relevance due to the negative effect of the fouling layer over the cycle performance.
- Proper design and operation of a heat exchanger must estimate the fouling resistance to the heat transfer.
- Reliable prediction of fouling formation allows for cost effective maintenance scheduling.
- Current predictive state of the art is mainly deterministic and is not accurate/reliable.
The data consists in measurements of the fouling thermal resistance with respect to time from two types of fouling:

- **Organic**: From a pilot plant that was arranged for side-stream monitoring in parallel with the real system at “Los Barrios” 550 MWe power plant (Bay of Algeciras, Southern tip of Spain).

- **Inorganic**: From a laboratory experiment in which artificial hard waters were recirculated through a steam heated annular test section for periods of 70 hours using stainless steel, copper and mild steel plain heat exchanger tubes and an externally-finned mild steel.

![Image of fouled condenser tubes]
Figure: Inorganic fouling for mild steel tubes at a fixed flow rate.
Hitting times (inorganic data)

Figure: **Left panel**: CDF of the hitting time for $B = 0.12$. **Right panel**: PDF of the hitting time to the critical level.

See poster by Marco Ballesio!
A factor in the failure of heavy-duty diesel engines is the wear of the cylinder liner.

Preventive maintenance to avoid failures can be carried out on the basis of the current condition of the liner.

Warranty clause: The liner should be substituted before it accumulates a wear level of 4.0 mm, in order to avoid expensive failures.

**Question:** when should we send the ship for maintenance?
The data set consists of wear levels observed on 32 cylinder liners of eight-cylinder SULZER engines and measured by a caliper with a precision of $\Delta = 0.05$ mm.

**Figure:** Due to the caliper’s finite precision, every single measurement of the wear process, $W(t)$, belongs to the lattice $\{0, \Delta, 2\Delta, \ldots\}$. 
Our solution

- A stochastic model of the evolution of the wear level.
- We obtained a complete probabilistic description of the evolution of the system at a reduced computational cost (sample-free).
- This allows us to compute any quantity of interest of the system, for example to identify when the system reaches some critical level.

This method is suitable to model the wear of any mechanical system in which observations are subject to a certain discrete precision.
Answer to the motivational question:

The ship should be sent to maintenance at the time that the thickness process, $X$, is less than or equal than $B = X_0 - 4mm$, where $X_0 = 5mm$ is the initial thickness.

Figure: **Left panel**: CDF of the hitting time for $B = 1$. **Right panel**: PDF of the hitting time to the critical level.

Given a critical level \( w_{\text{max}} \) which determines the residual lifetime \( \tau_{\text{max}} \) define

\[
R(t; 0, w_0) := P(\tau_{\text{max}} > t | W(0) = w_0).
\]

**Figure:** Behavior of the conditional residual reliability function, \( R(t; 0, w_0) \) for some values of \( w_0 \) (\( w_{\text{max}} = 4 \)). As expected, for a fixed residual lifetime \( t \), we have that \( R(t; 0, w_0) \) is a decreasing function of \( w_0 \).
Prediction of fatigue is very relevant in mechanical and structural engineering practice. Cost effective inspection intervals can be established based on such predictions.

Other works related to Inverse Problems

**Multilevel EnKF:** See Hakon Hoel’s talk today and related poster

**Design of Experiments:** See Marco Scavino’s talk today and related posters

**BIP for Heat Equation**