Adaptive Stochastic Galerkin FEM with Hierarchical Tensor Representations

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Stochastic model problem

Elliptic BVP with $u(x,y) \in \mathcal{V} := \mathcal{X} \otimes \mathcal{Y} := H_0^1(D) \otimes L_\pi^2(\Gamma)$

$$-\nabla \cdot (a(x,y)\nabla u(x,y)) = f(x) \quad \text{in } D \times \Gamma \quad \text{and} \quad u(x)|_{\partial D} = 0$$

on Lipschitz $D \subset \mathbb{R}^d$.

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on Lipschitz $D \subset \mathbb{R}^d$. Expansion of random field a

$$a(x,y) = a_0 + \sum_{m=1}^{\infty} a_m(x) y_m$$

with independent and uniformly distributed random variables

$$y = (y_m)_{m=1}^{\infty} \in \Gamma := [-1,1]^{\infty}, \qquad \pi = \bigotimes_{m=1}^{\infty} \pi_m$$

$$\text{(UEC)} \qquad \sum_{m=1}^{M} \left\| \frac{a_m}{a_0} \right\|_{L^\infty(D)} < 1, \quad a_0 > 0, \quad a_0, a_0^{-1} \in L^\infty(D)$$

Discretisation with (i) P_k conforming FEM in $\mathcal{X}$
and (ii) orthogonal polynomials in $\mathcal{Y}$

- $\longrightarrow$ very high dimensional coupled system (*curse of dimensionality* w.r.t. parameter dimensions)
- reduction of computational complexity necessary
- some approaches:
  - sampling or interpolation
  - a priori or a posteriori adaptivity
  - model order reduction (RBM, low-rank tensors)

Piecewise polynomials of degree p on mesh $\mathcal{T}$ of D

$\longrightarrow$ FEM space $\mathcal{X}_p(\mathcal{T}) := \operatorname{span}\{\varphi_i\}_{i=0}^{N-1} \subset \mathcal{X}$

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Tensorised Legendre polynomials

$$\mu \in \mathcal{F} := \{\mu \in \mathbb{N}_0^\infty : |\operatorname{supp}\mu| < \infty\} : \ P_\mu(y) := \prod_{m \in \operatorname{supp}\mu} P_{\mu_m}(y_m)$$

$\longrightarrow$ $(P_\mu)_{\mu\in\mathcal{F}}$ is orthonormal basis of $\mathcal{Y} := L^2_\pi(\Gamma) = \bigotimes_{m=1}^\infty L^2_{\pi_m}(\Gamma_m)$

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(Finite) index set $\Lambda \subset \mathcal{F}$ determines discrete approximation

$u_N \in \mathcal{V}_N := \mathcal{X}_p \otimes \operatorname{span}\{P_\mu\}_{\mu\in\Lambda} \subset \mathcal{V}$

$$\longrightarrow u_N(x,y) = \sum_{\mu\in\Lambda} u_{N,\mu}(x) P_\mu(y), \quad (u_{N,\mu})_{\mu\in\Lambda} \in \prod_{\mu\in\Lambda} \mathcal{X}_p(\mathcal{T})$$

where coefficients $u_{N,\mu}$ are s.t.

$$\mathbb{E}_\pi[\langle u_N, v\rangle_A] = \mathbb{E}_\pi[\langle f, v\rangle] \qquad (v \in \mathcal{V}_N)$$

Define operators

$$A = \sum_{m=0}^{\infty} A_m \quad \text{and} \quad A_m v := -\nabla \cdot (a_m \nabla v)$$

with

$$A : v \mapsto -\nabla \cdot (a(x, y)\nabla v).$$

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Expansion $u(x,y) = \sum_{\mu \in \mathcal{F}} u_\mu(x) P_\mu(y)$ holds with

$$A_0 u_\mu + \sum_{m=1}^{\infty} A_m (\beta_{\mu_m+1} u_{\mu+\epsilon_m} + \beta_{\mu_m} u_{\mu-\epsilon_m}) = f\delta_{\mu 0} \qquad (\mu \in \Lambda)$$

due to $y_m P_\mu(y) = \beta_{\mu_m+1} P_{\mu+\epsilon_m}(y) + \beta_{\mu_m} P_{\mu-\epsilon_m}(y)$.

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- single mesh higher-order FEM [EGSZ2]

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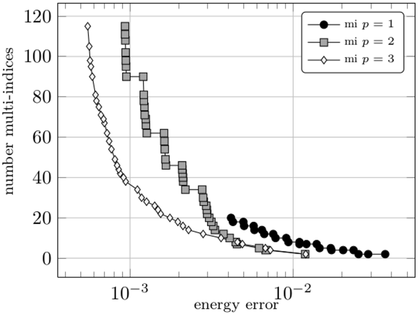

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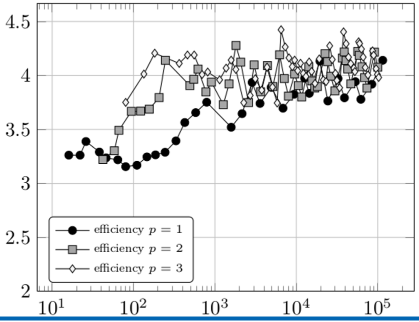
- operator construction *on the fly*
- separate meshes per stochastic dimension [EGSZ1]
- single mesh higher-order FEM [EGSZ2]
- guaranteed error control [EM]



Strategy B (compressed Λ)

$\longrightarrow$ Compress large $\Lambda \subset \mathcal{F}$ as efficiently as possible.

[1] but not specific multiindices

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- still choose set adaptively[1]
  - number dimensions M
  - anisotropic polynomial degrees $d_1, \ldots, d_M$
- however: employ full tensor set

$$\Lambda := \left\{(\mu_1, \ldots, \mu_M, 0, \ldots) \in \mathcal{F} \mid \mu_m = 0, \ldots, d_m - 1; m = 1, \ldots, M\right\}$$

which grows exponentially in M (curse of dimensionality)

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$\longrightarrow$ tensor structure of problem has to be exploited

[1] but not specific multiindices

Stochastic solution field can be written as

$$u_N(x,y) = \sum_{i=0}^{N-1} \sum_{\mu\in\Lambda} U(i,\mu)\varphi_i(x)P_\mu(y)$$

with coefficient tensor $U \in \mathbb{R}^{N\times d_1\times\cdots\times d_M}$ determined by

$$\mathbf{A}(U) := \left(\sum_{m=0}^{M} \mathbf{A}_m\right)(U) = F$$

where

$$A_m = K_m \otimes I \otimes \ldots \otimes B_m \otimes \ldots \otimes I$$
$$F = f \otimes e_1 \otimes \ldots \otimes e_1$$

Problem has a natural tensor structure

$$A_m(i,\mu,j,\nu) = \int_D a_m(x)\nabla\varphi_i \cdot \nabla\varphi_j \,\mathrm{d}x \int_{-1}^{1} P_{\mu_1}P_{\nu_1}\,\mathrm{d}y_1 \cdots$$
$$\cdots \int_{-1}^{1} y_m P_{\mu_m}P_{\nu_m}\,\mathrm{d}y_m \cdots \int_{-1}^{1} P_{\mu_M}P_{\nu_M}\,\mathrm{d}y_M$$

$\longrightarrow$ high-order tensor representations

$$A_m = K_m \otimes I \otimes \ldots \otimes B_m \otimes \ldots \otimes I$$

with

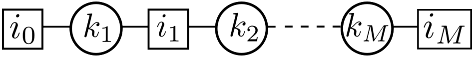
$$K_m(i,j) = \int_D a_m(x)\nabla\varphi_i \cdot \nabla\varphi_j \,\mathrm{d}x$$
$$B_m(\mu_m,\nu_m) = \int_{-1}^{1} y_m P_{\mu_m}P_{\nu_m}\,\mathrm{d}y_m$$

For the tensor

$$U \in \mathbb{R}^{N \times d_1 \times \ldots \times d_M}, \quad U(i,\mu) = \mathcal{U}(i_0, i_1, \ldots, i_M)$$

we obtain the TT representation (SVD separated decomposition)

$$\mathcal{U}(i_0, \underbrace{i_1, \ldots, i_M}_{=\mu}) = \sum_{k_1=1}^{r_1} \cdots \sum_{k_M=1}^{r_M} U_0(i_0,k_1) U_1(k_1,i_1,k_2) \cdots U_M(k_M,i_M)$$



$$= \mathrm{U}_0(i_0)\mathrm{U}_1(i_1)\cdots \mathrm{U}_M(i_M)$$

i.e. a “separation of variables” with $U_m \in \mathbb{R}^{r_m \times d_m \times r_{m+1}}$ and $\mathrm{U}_m(i_m) \in \mathbb{R}^{r_m \times r_{m+1}}$.

For $u_N = \sum_{i=0}^{N-1} \sum_{\mu\in\Lambda} U(i,\mu)\varphi_i P_\mu \in \mathcal{V}_N$,

$$U(i,\mu_1,\dots,\mu_M) = \sum_{k_1=1}^{r_1} \cdots \sum_{k_{M+1}=1}^{r_{M+1}} U_0(i,k_1) \prod_{m=1}^{M} U_m(k_m,\mu_m,k_{m+1}).$$

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Expectation is determined by

$$\begin{aligned}
\mathbb{E}(u_N) &= \int_\Gamma u_N(y)\,\mathrm{d}\pi(y) \\
&= \sum_{i=0}^{N-1} \sum_{\mu\in\Lambda} U(i,\mu)\varphi_i \prod_{m=1}^{M} \int_{-1}^{1} P_{\mu_m}(y_m)\,\mathrm{d}\pi(y_m) \\
&= \sum_{i=0}^{N-1} U(i,1,\dots,1)\varphi_i.
\end{aligned}$$

Thm. [EGSZ]: Energy error bound w.r.t. some $w_N \in \mathcal{V}_p(\Lambda;\mathcal{T})$ and Galerkin approximation $u_N \in \mathcal{V}_p(\Lambda;\mathcal{T})$ of exact solution u

$$\|w_N - u\|_{\mathcal{A}} \lesssim \underbrace{\eta(w_N,\Lambda,\mathcal{T})}_{\text{approximation}} + \underbrace{\zeta(w_N,\partial\Lambda)}_{\text{tail}} + \underbrace{\|w_N - u_N\|_{\mathcal{A}}}_{\text{consistency}}$$

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with estimators

$$\eta_\mu(w_N)^2 := \sum_{T\in\mathcal{T}} \eta_{\mu,T}(w_N)^2 + \sum_{S\in\mathcal{S}} \eta_{\mu,S}(w_N)^2$$

$$\eta_{\mu,T}(w_N) := h_T \|a_0^{-1/2}(f\delta_{\mu 0} + \nabla\cdot\sigma_\mu(w_N))\|_{L^2(T)}$$

$$\eta_{\mu,S}(w_N) := h_S^{1/2} \|a_0^{-1/2}[\![\sigma_\mu(w_N)]\!]_S\|_{L^2(S)}$$

$$\zeta_\nu(w_N) := \sum_{m=1}^{\infty} \left\|\frac{a_m}{a_0}\right\|_{L^\infty(D)} \left(\beta_{\nu_m+1}\left\|w_{N,\nu+\epsilon_m}\right\|_{\mathcal{X}} + \beta_{\nu_m}\left\|w_{N,\nu-\epsilon_m}\right\|_{\mathcal{X}}\right)$$

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$$\|w_N - u\|_{\mathcal{A}} \lesssim \eta(w_N,\Lambda,\mathcal{T}) + \zeta(w_N,\partial\Lambda) + \|w_N - u_N\|_{\mathcal{A}}$$

adaptive algorithm (single iteration)

- evaluate solution w_N in TT format by ALS
- evaluate error bounds $\eta(u_N,\Lambda,\mathcal{T})$ and $\zeta(u_N,\partial\Lambda)$
- spatial refinement if η dominates $\longrightarrow$ refine mesh $\mathcal{T}$
- stochastic refinement if ζ dominates $\longrightarrow$ enlarge active set Λ
- tensor upgrade if consistency error dominates
  $\longrightarrow$ increase TT rank

All estimators (approximation, tail, algebraic) can be evaluated efficiently in TT format, e.g.,

$$
\begin{aligned}
\sum_{\mu\in\Lambda} \eta_{\mu,T}(w_N)^2 := \sum_{\mu\in\Lambda} h_T^2 \| a_0^{-1/2}(f\delta_{\mu 0} + \nabla\cdot\sigma_\mu(w_N))\|^2_{L^2(T)} \\
= \sum_{\mu\in\Lambda} h_T^2 \langle f\delta_{\mu 0} + \nabla\cdot\sigma_\mu(w_N), a_0^{-1}(f\delta_{\mu 0} + \nabla\cdot\sigma_\mu(w_N))\rangle_T \\
= h_T^2 (\langle f\delta_{\mu 0}, a_0^{-1} f\rangle_T + 2\langle f\delta_{\mu 0}, a_0^{-1}\nabla\cdot\sigma_0(w_N)\rangle_T \\
+ \sum_{\mu\in\Lambda} \langle \nabla\cdot\sigma_\mu(w_N), a_0^{-1}\nabla\cdot\sigma_\mu(w_N)\rangle_T)
\end{aligned}
$$

for $w_N = \sum_{i=0}^{N-1} \sum_{\mu\in\Lambda} W(i,\mu)\varphi_i P_\mu \in \mathcal{V}_N(\Lambda;\mathcal{T})$ with numerical flux

$$
\sigma_\mu(w_N) = a_0 \nabla w_{N,\mu} + \sum_{m=1}^{M} a_m \nabla(\beta_{\mu_m+1} w_{N,\mu+\epsilon_m} + \beta_{\mu_m} w_{N,w_\mu-\epsilon_m})
$$

On any $T \in \mathcal{T}$,

$$\sum_{\mu \in \Lambda} \langle \nabla \cdot \sigma_\mu(w_N), a_0^{-1} \nabla \cdot \sigma_\mu(w_N) \rangle$$

$$= \sum_{m_1,m_2=0}^{M} \sum_{i,j=0}^{N-1} \langle \nabla \cdot (a_{m_1} \nabla \varphi_i), a_0^{-1} \nabla \cdot (a_{m-2} \nabla \varphi_j) \rangle_T \sum_{\mu \in \Lambda} W^{m_1}(i,\mu) W^{m_2}(j,\mu)$$

with TT tensors $W^m := [I_0 \otimes I_1 \otimes \cdots \otimes B_m \otimes \cdots \otimes I_M](W)$.

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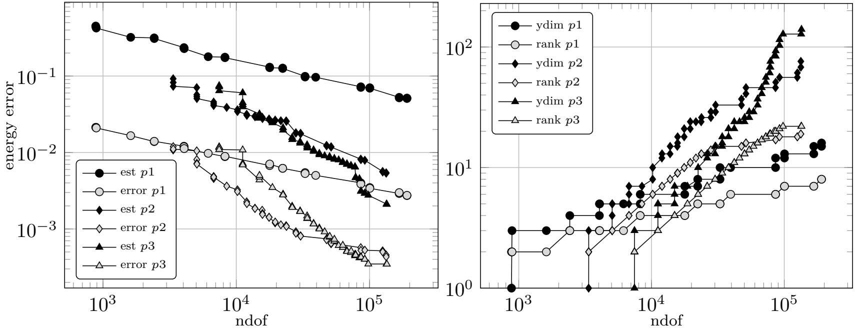
with TT tensors $W^m := [I_0 \otimes I_1 \otimes \cdots \otimes B_m \otimes \cdots \otimes I_M](W)$.
Another low-rank decomposition yields

$$= \sum_{m_1,m_2=0}^{M} \sum_{k_1=1}^{r_1} \langle \nabla \cdot (a_{m_1} \sum_{i=0}^{N-1} \nabla \varphi_i V^{m_1,m_2}(i,k_1)), a_0^{-1} \nabla \cdot (a_{m_2} \sum_{j=0}^{N-1} \nabla \varphi_j) W_0(j,k_1) \rangle_T.$$

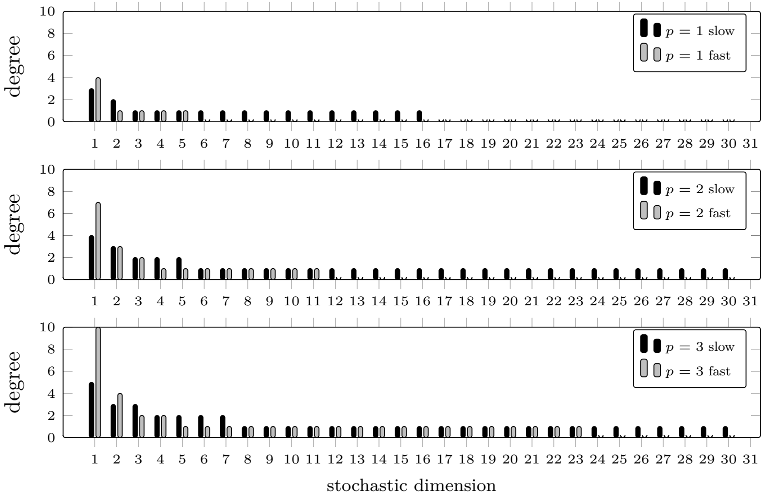
All other estimator terms can be treated similarly.



(estimator, energy error (left) and stochastic dimensions, TT rank (right))

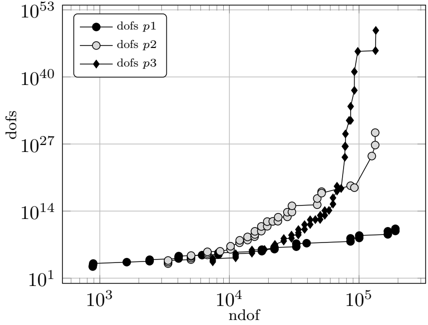
With random coefficient $a(x,y) := a_0 + \sum_{m=1}^{\infty} a_m(x) y_m$

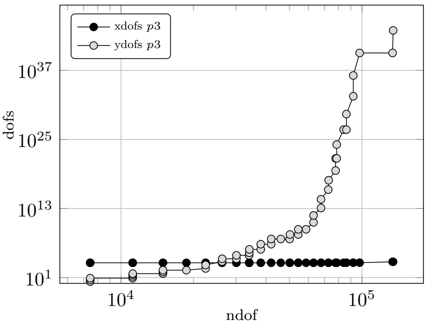
$$a_m(x) := \alpha m^{-2} \cos(2\pi\beta_1(m)x_1) \cos(2\pi\beta_2(m)x_2)$$



(anisotropic gpc degrees per stochastic dimension)

Stationary random diffusion on square domain





(full overall problem size (left) and dofs of $\mathcal{X}$ and $\mathcal{Y}$ (right))

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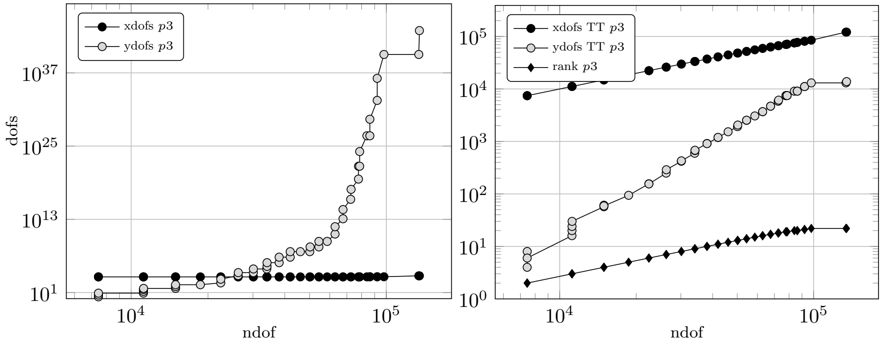

(dofs of $\mathcal{X}$ and $\mathcal{Y}$ (left) and in TT format (right))

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