Static models, recursive estimators and the zero-variance approach

G. Rubino

4th SRI UQ Workshop

Outline

The problem

- It belongs to an important class of static models used in many engineering areas.
- Consider a multi-component system composed of M independent components.
- Components and system are in a random "state" belonging to the set $\{\text{up, down}\}$. The state of component i is denoted by the Binary r.v. X_i coding 1 for "up" and 0 for "down".
- The random vector $X = (X_1, \ldots, X_M)$ is called the *system's configuration*.
- The system's state is given by the *structure function* Φ from $\{0, 1\}^M$ into $\{0, 1\}$, where $\Phi(x) = 1(\text{sys. works when config. is } x)$. For instance, for a series of two components, $\Phi(x_1, x_2) = x_1 x_2$.

- We are given Φ and the M numbers $r_1, \dots, r_M$ where $r_i = \mathbb{P}(X_i = 1)$. The goal is to find $R = \mathbb{P}(\Phi(X) = 1) = \mathbb{E}(\Phi(X))$.
- Equivalently, we can work with the numbers $q_1, \dots, q_M$ where $q_i = 1 - r_i$ and look for $Q = 1 - R$.
- The number r_i (resp. q_i) is the (*elementary*) *reliability* (resp. the (*elementary*) *unreliability*) of component i.
- The law of X is $\pi(x) = \prod_{i:\ x_i=1} r_i \prod_{j:\ x_j=0} q_j$.
- The number R is the reliability of the system, and Q is its unreliability. See that we have discrete sums here, but possibly with a huge number of terms:

$$R = \sum_{x\in\{0,1\}^M} \Phi(x)\pi(x), \quad Q = \sum_{x\in\{0,1\}^M} \Big[1 - \Phi(x)\Big]\pi(x).$$

- Exactly computing R or Q leads to NP-hard problems. This means here that we can not analyze exactly models even with moderate sizes.
- In the Monte Carlo case, the problem to deal with is the rare event situation, where $R \approx 1$ ($Q \approx 0$).

Reference problem

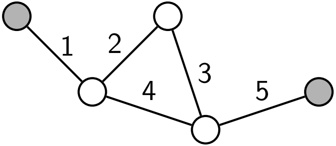
- Network reliability: a reference problem in this family (and an active research area for years).
- We are typically given an undirected connected graph $\mathcal{G}$ without loops, whose M edges represent the system's components.
- With each configuration x we associate the partial graph $\mathcal{G}(x)$ build by removing any edge i of $\mathcal{G}$ for which $x_i = 0$.
- A subset of nodes (called *terminals*), denoted here by K, is selected, and the structure function is

$$\Phi(x) = 1\big(\text{the nodes in } K \text{ are connected in } \mathcal{G}(x)\big).$$

- In other words, instead of giving a "table" Φ with 2^M entries, we give a graph with M edges, and Φ becomes implicitly defined.

Example 1

- For instance, let $\mathcal{G}$ be the graph



where K is the set composed of the two grey nodes.
- We have

$$R = r_1\big[1-(1-r_4)(1-r_2r_3)\big]r_5 = r_1(r_2r_3+r_4-r_2r_3r_4)r_5$$

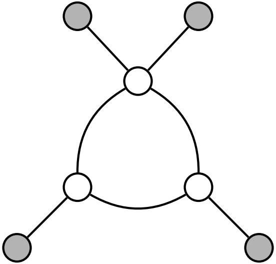
or, equivalently, the dual expresion

$$Q = Q_a + Q_b - Q_aQ_b,$$

with $Q_a = q_1 + q_5 - q_1q_5$ and $Q_b = q_4(q_2 + q_3 - q_2q_3)$.

Example 2

Consider this other example representing a typical communication network, with K being the set of terminal machines (the grey nodes) connected through a *backbone* (the ring of white nodes).



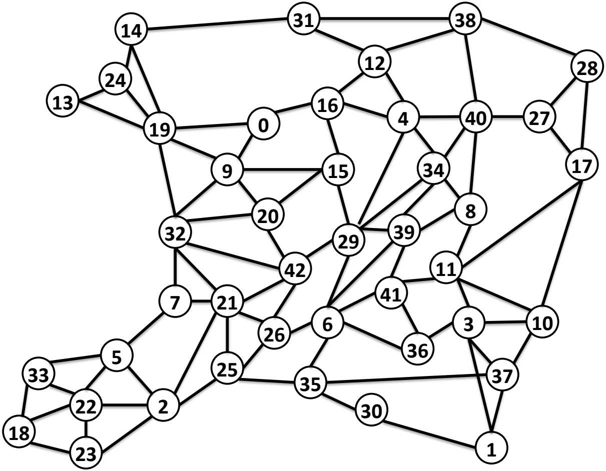
If r is the elementary reliability of any node, then

$$R = r^4\big(r^3 + 3r^2(1-r)\big) = r^6(3-2r).$$

Complexity issues

- As stated before, and simplifying, computing R is NP-hard.
- This is still the case even if we restrict the models to the family of planar graphs having degree at most three at any node.
- Most "moderate sized" models are thus "out-of-reach" (see examples below, from the scientific literature).
- Only available way to analyze them: Monte Carlo.
- Main problem then: rare events.

Example of “out-of-reach” network

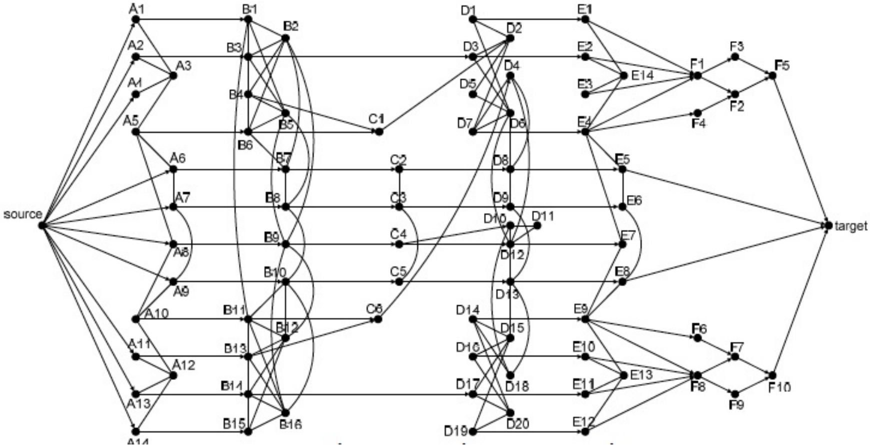


European optical comm. infrastructure (41 nodes, 180 links)

- A careful examination of this topology shows that there is (almost) no special regularity that can be exploited to diminish the complexity of an exact analysis.
- So, whatever the choice of the terminals, no hope to know R in this example (even after many years of hardware evolution).
- Standard Monte Carlo can deal with these sizes and with much larger models ... except if R is too close to 1.

Another “out-of-reach” case

- One of many models used in the analysis of the Boeing Dreamliner 787 aircraft:



- 82 nodes, 171 links

This work

- I will describe the main ideas in our 2015 paper "*Balanced and Approximate Zero-Variance Recursive Estimators for the Static Communication Network Reliability Problem*", TOMACS, 25(1): 5:1-5:19, 2015, co-authored with H. Cancela, M. El Khadiri and B.Tuffin.
- It consists of a new approach combining a powerful recursive estimation technique with an Importance Sampling scheme approximating the zero-variance one.

Outline

Monte Carlo

- Denote $Y = 1 - \Phi(X)$, and let $Y^{(1)}, Y^{(2)}, \ldots, Y^{(n)}$ be n independent copies of Y. Our target is $Q = \mathbb{E}(Y)$.
- Convenient notation from here: $Q = q(\mathcal{G})$, $R = r(\mathcal{G})$.
- The standard estimator of Q is $Y_{SMC} = n^{-1}(Y^{(1)} + Y^{(2)} + \cdots + Y^{(n)})$.
- In the rare event case, what matters is the relative error RE, which is captured here by $\sqrt{\mathbb{V}(Y_{SMC})} = \sqrt{RQ/n}$:

$$RE = c\frac{\sqrt{r(\mathcal{G})}}{\sqrt{nq(\mathcal{G})}},$$

and we see that $RE \to \infty$ as $q(\mathcal{G}) \to 0$.

Bounded Relative Error

- An unbiased estimator Y' of $\mathbb{E}[Y]$ has *Bounded Relative Error* (BRE) if *RE* remains bounded as the event becomes rarer.
- Formally, we have BRE if $\sqrt{\mathbb{V}(Y')}/\mathbb{E}[Y]$ seen as a family of functions indexed by the sample size n, is uniformly bounded when $\mathbb{E}[Y] \to 0$, or equivalently, if $\mathbb{E}\left(Y'^2\right)/\mathbb{E}^2(Y)$ is uniformly bounded as $\mathbb{E}[Y] \to 0$.
- BRE implies that the sample size required to get a given relative error is not sensitive to the rarity of the event.
- The standard estimator does not possess this property.

Other relevant properties of estimators

- **Weaker than BRE**. An unbiased estimator Y' of $\mathbb{E}[Y]$ is *Asymptotically Optimal* or *Logarithmically Efficient* if
$$\lim_{\mathbb{E}(Y) \to 0} \frac{\ln \mathbb{E}\big(Y'^2\big)}{\ln \mathbb{E}(Y)} = 2.$$
- **Stronger than BRE**. An unbiased estimator Y' of $\mathbb{E}[Y]$ verifies the *Vanishing Relative Error* (VRE) property if $\sqrt{\mathbb{V}(Y')}/\mathbb{E}[Y] \to 0$ as $\mathbb{E}[Y] \to 0$.

Outline

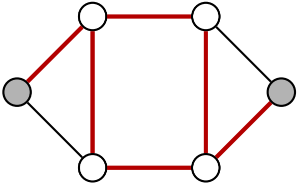
Cuts

Cuts in a structure function (or in a graph setting, to simplify):

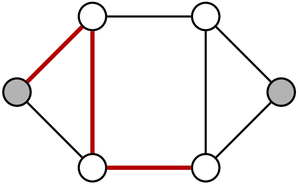
- A *cut* is a set of components such that if they are all down, the system is down.
- A *mincut* is a cut that has no strict subset that is also a cut.

Examples of cuts

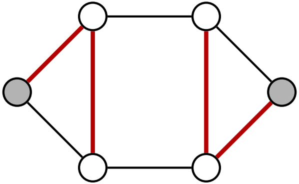
A cut but not a mincut:



A mincut:

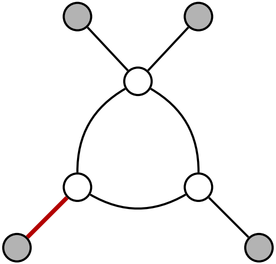


Another mincut:

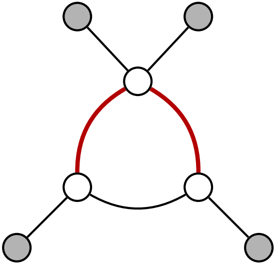


More examples of cuts

A mincut:



Another mincut:



Recursive Variance Reduction (RVR)

- Principle: select a cutset, i.e., a set $\mathcal{C}$ of links whose failure ensures the system failure.
- Denote by $q_\mathcal{C}$ the probability that all links in $\mathcal{C}$ are failed, that is, $q_\mathcal{C} = \prod_{i\in\mathcal{C}} q_i$.
- By definition, if all links in $\mathcal{C}$ are failed, the system is failed. Consequently, $q_\mathcal{C} \le Q$.
- Put some order on the links of $\mathcal{C}$. Let's denote $\mathcal{C} = \{1, 2, \ldots\}$.
- $B_j =$ "the $j-1$ first links of $\mathcal{C}$ are down, but the jth is up".
- $\mathbb{P}\big[B_j\big] = \big(\prod_{k=1}^{j-1} q_k\big) r_j$.
- Define $p_j = \mathbb{P}\big[B_j \mid \text{at least one link of } \mathcal{C} \text{ is working}\big] = \mathbb{P}\big[B_j\big]/(1 - q_\mathcal{C})$.

Recursive Variance Reduction (RVR)

The RVR estimator:

- Select a cutset, and compute $q_{\mathcal{C}}$ and the p_js.
- Pick an edge at random in $\mathcal{C}$ according to the probability distribution $(p_j)_{j=1,\cdots,|\mathcal{C}|}$.
- Let the chosen edge be the jth. Call $\mathcal{G}_j$ the graph obtained from $\mathcal{G}$ by deleting the first $j-1$ edges of $\mathcal{C}$ and by contracting the jth.
- The value y_{RVR} returned by the RVR estimator of $q(\mathcal{G})$, the unreliability of $\mathcal{G}$, is recursively defined as

$$y_{RVR}(\mathcal{G}) = q_{\mathcal{C}} + (1 - q_{\mathcal{C}})y_{RVR}(\mathcal{G}_j).$$

RVR estimator

Formally, the RVR estimator of $Q = q(\mathcal{G})$ is the random variable

$$Y_{RVR} = q_{\mathcal{C}} + (1 - q_{\mathcal{C}}) \sum_{j=1}^{|\mathcal{C}|} \frac{\mathbf{1}_{B_j}}{1 - q_{\mathcal{C}}} Y_{RVR}(\mathcal{G}_j).$$

Theorem

The estimator is unbiased: $\mathbb{E}[Y_{RVR}] = q(\mathcal{G}) = Q$.

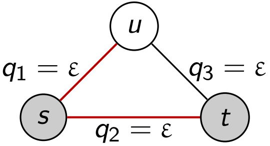
Proof: induction using the recursion.

Second moment is

$$\mathbb{E}[Y_{RVR}^2] = q_{\mathcal{C}}^2 + 2q_{\mathcal{C}}(1 - q_{\mathcal{C}}) \left(\sum_{j=1}^{|\mathcal{C}|} \frac{\mathbb{P}[B_j]}{1 - q_{\mathcal{C}}} \mathbb{E}[Y_{RVR}(\mathcal{G}_j)] \right) + (1 - q_{\mathcal{C}})^2 \left(\sum_{j=1}^{|\mathcal{C}|} \frac{\mathbb{P}[B_j]}{1 - q_{\mathcal{C}}} \mathbb{E}[Y_{RVR}^2(\mathcal{G}_j)] \right).$$

No Bounded Relative Error for RVR

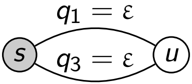
The RVR algorithm does not verify the Bounded Relative Error property. Consider the example

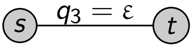


- Selected cut: the two red links, ordering them as first the link from s to t.
- $q_{\mathcal{C}} = \varepsilon^2$.

$$
\begin{aligned}
\mathbb{E}[Y_{RVR}^2] &= \varepsilon^4 + 2\varepsilon^2\Big[(1-\varepsilon)\mathbb{E}[Y_{RVR}(\mathcal{G}_1)] + \varepsilon(1-\varepsilon)\mathbb{E}[Y_{RVR}(\mathcal{G}_2)]\Big] \\
&+ (1-\varepsilon^2)\Big[(1-\varepsilon)\mathbb{E}[Y_{RVR}^2(\mathcal{G}_1)] + \varepsilon(1-\varepsilon)\mathbb{E}[Y_{RVR}^2(\mathcal{G}_2)]\Big].
\end{aligned}
$$

where

- $\mathcal{G}_1$: link from s to t is working $\rightsquigarrow$ s and t are merged (the system is necessarily connected).


$Y_{RVR}(\mathcal{G}_1) = 0$. Thus $\mathbb{E}[Y_{RVR}(\mathcal{G}_1)] = \mathbb{E}[Y^2_{RVR}(\mathcal{G}_1)] = 0$.
- $\mathcal{G}_2$: link from s to t failed, but the one from s to u is working $\rightsquigarrow$ s and u are merged.


$\mathbb{E}[Y_{RVR}(\mathcal{G}_2)] = \varepsilon$, $\mathbb{E}[Y^2_{RVR}(\mathcal{G}_2)] = \varepsilon^2$.
- Finally, $\mathbb{E}[Y^2_{RVR}] = \Theta(\varepsilon^3)$, and $\mathbb{E}[Y^2_{RVR}]/(\mathbb{E}[Y_{RVR}])^2 = \Theta(\varepsilon^{-1}) \to \infty$ as $\varepsilon \to 0$.

Observe that we may have BRE or not, depending on the ordering of the links.

Outline

Balanced RVR

- Non-BRE comes from the crude distribution for sampling the first working link on the cut.
- *Importance Sampling (IS)* used instead; that is, the sampling distribution of the first line up in the cut is not anymore (p_j).
- So far, we built a partition by assigning to the events B_j, for $1 \leq j \leq |\mathcal{C}|$, the conditional probabilities

$$p_j = \mathbb{P}\big[B_j \mid \text{at least one link of } \mathcal{C} \text{ is working}\big].$$

- Let us write the RVR estimator as

$$Y_{RVR} = q_{\mathcal{C}} + (1 - q_{\mathcal{C}}) \sum_{j=1}^{|\mathcal{C}|} \mathbf{1}_{B'_j} Y_{RVR}(\mathcal{G}_j),$$

where B'_j represents the same event as B_j but with the (conditional) probability p_j.

Balanced RVR (BRD)

- Now, we change this probability p_j by the uniform distribution on $\{1, 2, \cdots, |\mathcal{C}|\}$, $\tilde{p}_j = 1/|\mathcal{C}|$, for event B_j'.
- Let us call Y_{BRD} (BRD: Balance Recursive Decomposition) the corresponding estimator. Using this uniform distribution and the likelihood ratio $p_j/\tilde{p}_j$ to keep the estimator unbiased, we formally write

$$\begin{aligned} Y_{BRD} &= q_{\mathcal{C}} + (1 - q_{\mathcal{C}}) \sum_{j=1}^{|\mathcal{C}|} \mathbf{1}_{B_j'} \frac{p_j}{\tilde{p}_j} Y_{BRD}(\mathcal{G}_j) \\ &= q_{\mathcal{C}} + |\mathcal{C}| \sum_{j=1}^{|\mathcal{C}|} \mathbf{1}_{B_j'} \mathbb{P}[B_j] Y_{BRD}(\mathcal{G}_j). \end{aligned}$$

Results on Balanced RVR

Analyzing the relative error, we obtain

Theorem

The estimator Y_{BRD} is unbiased: $\mathbb{E}[Y_{BRD}] = Q$.
The BRD algorithm verifies the Bounded Relative Error property.

Proof: induction from the recursion, in particular, for the second claim, from

$$\begin{aligned}\mathbb{E}[Y_{BRD}^2] &= q_{\mathcal{C}}^2 + 2q_{\mathcal{C}}|\mathcal{C}|\left(\sum_{j=1}^{|\mathcal{C}|}\mathbb{P}[B_j]\mathbb{E}[Y_{BRD}(\mathcal{G}_j)]\right) \\ &\quad + |\mathcal{C}|^2\left(\sum_{j=1}^{|\mathcal{C}|}(\mathbb{P}[B_j])^2\mathbb{E}[Y_{BRD}^2(\mathcal{G}_j)]\right).\end{aligned}$$

Remarks on BRD

- Intuition behind BRD: make sure that probability of each event B_j' is $\Theta(1)$, so that no event is rare under IS.
  As a consequence, the probability $\mathbb{P}(B_j)$ is squared in the likelihood ratio (which was not the case for RVR), and BRE can be obtained.
- Note that any choice of distribution such that probability of each B_j' is $\Theta(1)$, leads to BRE as well.
- For some network topologies and link unreliability values, SMC has lower variance than BRD. Thus, the BRD estimator does not guarantee variance reduction in all contexts.

Outline

Intermezzo

- Move for a second to Markov chains.
- Consider a Markov chain X on some discrete state space S, with transition probability matrix (t.p.m.) P, and a cost (or reward) function c defined on the transitions; for instance, $c\colon S^2 \to \mathbb{R}_{\geq 0}$.
- We are interested in the mean cumulated cost up to some stopping time τ, $\mu = \mathbb{E}(Y)$, where $Y = \sum_{j=1}^{\tau} c(X_{j-1}, X_j)$.
- To estimate μ, suppose we use Importance Sampling (IS), with the change of measure given by the new t.p.m. P':

$$P'_{i,j} = \frac{P_{i,j}\left[c(i,j) + \mu_j\right]}{\sum_{k \in S} P_{i,k}\left[c(i,k) + \mu_k\right]},$$

where $\mu_h = \mathbb{E}(Y \mid X_0 = h)$.

- It can be proved that this change of measure is the zero-variance one (one run, exact value at the output), so, useless in practice because we need the initial target to implement it.
- "Zero-variance idea": use the new dynamics

$$P^*_{i,j} = \frac{P_{i,j}\left[c(i,j) + \mu^*_j\right]}{\sum_{k\in S} P_{i,k}\left[c(i,k) + \mu^*_k\right]},$$

  where μ^*_h is some (any) approximation to μ_h, even a poor one.
- This is one of the hottest topics in the area; in general, the method leads to very good (efficient) results.
- The heart of the method lies in the problem of finding an approximation μ^* to μ that can be computed very fast.
- In the paper, we managed to use the same idea in spite of the fact that we are in a static context.

Zero-variance IS

- In this context, the (exact) *zero-variance change of measure* chooses the best possible IS scheme for the first working link on the cut, that is, samples B'_j with the probability $\widetilde{p}_j$, with

$$\widetilde{p}_j = \frac{\mathbb{P}[B_j]q(\mathcal{G}_j)}{\sum_{k=1}^{|\mathcal{C}|}\mathbb{P}[B_k]q(\mathcal{G}_k)} \qquad (1)$$

- Resulting estimator:

$$Y_{ZVD} = q_{\mathcal{C}} + \left(\sum_{k=1}^{|\mathcal{C}|}\mathbb{P}[B_k]q(\mathcal{G}_k)\right)\sum_{j=1}^{|\mathcal{C}|}\mathbf{1}_{B'_j(\mathcal{G})}\frac{1}{q(\mathcal{G}_j)}Y_{ZVD}(\mathcal{G}_j).$$

Theorem

Y_{ZVD} has variance $\mathrm{Var}[Y_{ZVD}] = 0$.

Proof: induction using the recursions.

- Implementing this requires the knowledge of the $q(\mathcal{G}_i)$, but in that case, no need to simulate!
- Idea: use instead some (any) approximation $\hat{q}(\mathcal{G}_i)$ of $q(\mathcal{G}_i)$ plugged into (1). This gives a new estimator called Y_{AZVRD}:

$$Y_{AZVRD} = q_{\mathcal{C}} + \left(\sum_{k=1}^{|\mathcal{C}|} \mathbb{P}[B_k]\hat{q}(\mathcal{G}_k)\right) \sum_{j=1}^{|\mathcal{C}|} \mathbf{1}_{B'_j(\mathcal{G})} \frac{1}{\hat{q}(\mathcal{G}_j)} Y_{AZVRD}(\mathcal{G}_j).$$

Proposition

If for $1 \leq j \leq |\mathcal{C}|$, $\hat{q}(\mathcal{G}_j) = \Theta(q(\mathcal{G}_j))$ as $q(\mathcal{G}) \to 0$, Y_{AZVRD} verifies the BRE property.

Approx. 0-var. Recursive Decomposition

Define the *mincut-maxprob* approximation $\hat{q}(\mathcal{G})$ of $q(\mathcal{G})$ as the maximal probability of a mincut of graph $\mathcal{G}$ (can be computed in polynomial time).

Proposition

With the mincut-maxprob approximation, $\hat{q}(\mathcal{G}_j) = \Theta(q(\mathcal{G}_j))$ as $q(\mathcal{G}) \to 0$; therefore the BRE property is obtained.

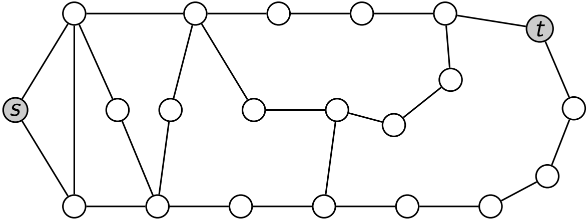
Proposition

If, $\hat{q}(\mathcal{G}_j) = q(\mathcal{G}_j) + o(q(\mathcal{G}_j))$ as $q(\mathcal{G}) \to 0$ for all $1 \le j \le |\mathcal{C}|$, the Vanishing relative (VRE) property (RE $\to 0$ as $q(\mathcal{G}) \to 0$, much stronger than just being bounded) is verified.

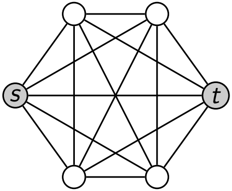
Outline

Topologies: arpanet, complete graphs,

arpanet:

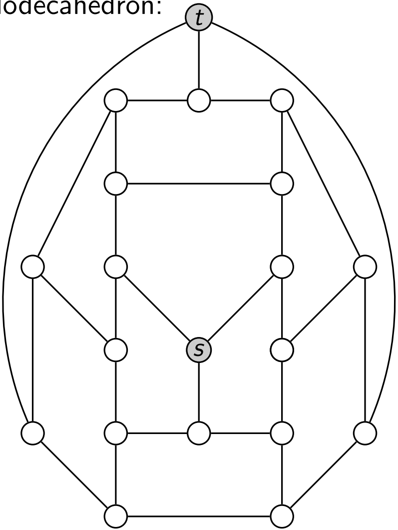


C_6:

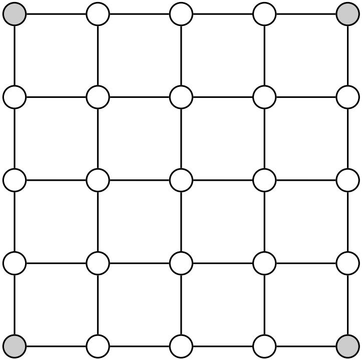


... dodecahedron and grids

dodecahedron:



grid 5:



Comparisons

The (normalized) relative error[1] for various methods and unreliabilities ε of links (homogeneous case), on the dodecahedron topology

Method	$\varepsilon = 0.1$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$
[F]	2.6 e+00	1.3 e+00	4.3 e−01	1.4 e−02
[S1]	4.0 e+00	6.2 e+00	7.7 e+00	8.9 e+00
[S2]	4.6 e+00	7.1 e+00	8.6 e+00	8.8 e+00
[B]	3.0 e+00	4.2 e+00	4.3 e+00	4.4 e+00
[Z]	1.2 e+00	1.7 e−01	5.7 e−02	1.7 e−02
[R]	8.4 e−01	7.1 e−01	7.1 e−01	7.1 e−01
BRD [A]	9.5 e−01	7.0 e-01	7.1 e−01	7.1 e−01
AZVRD [A]	2.8 e−01	5.1 e−02	1.6 e−02	5.0 e−03

[1] Denoting RE_N the relative error, we use here $\sqrt{N} \cdot RE_N/z$, with, say, $z = 1.96$.

F G. S. Fishman. 1986. A Monte Carlo sampling plan for estimating network reliability. Operations Research 34, 4, 581–594 (method based on bounds).

S1 Z. I. Botev, P. L'Ecuyer, G. Rubino, R. Simard and B. Tuffin. 2013. Static network reliability estimation via generalized splitting. INFORMS Journal on Computing 25, 1, 56–71 (a generalization of splitting).

S2 L. Murray, H. Cancela, and G. Rubino. 2013. A splitting algorithm for network reliability. IIE Transactions 45, 2, 177–189 (another adaptation of splitting to static problems).

B I. B. Gertsbakh and Y. Shpungin. 2010. Models of Network Reliability. CRC Press, Boca Raton, FL, US (the so-called turnip method).

Z P. L'Ecuyer, G. Rubino, S. Saggadi and B. Tuffin. 2011. Approximate zero-variance importance sampling for static network reliability estimation. IEEE Transactions on Reliability 8, 4, 590–604 (another zero-variance approximation).

R H. Cancela and M. El Khadiri. 1995. A recursive variance-reduction algorithm for estimating communication-network reliability. IEEE Transactions on Reliability 44, 4, 595–602 (the original RVR technique).

A H. Cancela, M. El Khadiri, G. Rubino and B. Tuffin. 2015. Balanced and Approximate Zero-Variance Recursive Estimators for the Static Communication Network Reliability Problem. TOMACS, 25(1): 5:1–5:19 (our papers' methods).

Illustration of the BRE and VRE properties

Network	ε	$q(\mathcal{G})$	$\frac{\sqrt{n}\times RE_{SMC}}{c_\alpha}$	$\frac{\sqrt{n}\times RE_{RVR}}{c_\alpha}$	$\frac{\sqrt{n}\times RE_{BRD}}{c_\alpha}$	$\frac{\sqrt{n}\times RE_{AZVRD}}{c_\alpha}$
Arpanet	5 e−01	9.6398994 e−01	1.93 e−01	6.33 e−02	4.16 e−01	4.27 e−01
Arpanet	3 e−01	6.8150724 e−01	6.84 e−01	3.20 e−01	1.10 e+00	1.35 e+00
Arpanet	1 e−01	9.5422918 e−02	3.08 e+00	1.27 e+00	2.01 e+00	3.24 e+00
Arpanet	1 e−03	6.0558106 e−06	4.06 e+02	2.09 e+01	1.24 e+00	9.67 e−01
Arpanet	1 e−05	6.0005600 e−10	4.08 e+04	2.11 e+02	1.26 e+00	9.82 e−02
Dod	5 e−01	7.0974499 e−01	6.39 e−01	1.77 e−01	9.17 e−01	5.17 e−01
Dod	3 e−01	1.6851806 e−01	2.22 e+00	5.70 e−01	1.93 e+00	7.70 e−01
Dod	1 e−01	2.8796013 e−03	1.86 e+01	8.37 e−01	9.53 e−01	2.76 e−01
Dod	1 e−03	2.0060181 e−09	2.23 e+04	7.08 e−01	7.06 e−01	1.59 e−02
Dod	1 e−05	2.0000600 e−15	2.24 e+07	7.07 e−01	7.07 e−01	1.58 e−03
Grid5	5 e−01	9.6062484 e−01	2.02 e−01	2.66 e−02	4.55 e−01	1.01 e−01
Grid5	3 e−01	5.2094890 e−01	9.59 e−01	1.53 e−01	1.17 e+00	2.29 e−01
Grid5	1 e−01	4.8160510 e−02	4.45 e+00	1.40 e−01	1.09 e+00	1.35 e−01
Grid5	1 e−03	4.0080020 e−06	4.99 e+02	1.58 e−02	1.14 e+00	1.37 e−02
Grid5	1 e−05	4.0000800 e−10	5.00 e+04	1.58 e−03	1.15 e+00	1.37 e−03
C6	5 e−01	7.6416016 e−02	3.48 e+00	1.15 e−01	3.43 e−01	1.12 e−01
C6	3 e−01	5.2672775 e−03	1.37 e+01	9.61 e−02	5.32 e−01	9.06 e−02
C6	1 e−01	2.0076587 e−05	2.23 e+02	1.78 e−02	7.53 e−01	1.71 e−02
C6	1 e−03	2.0000000 e−15	2.24 e+07	1.58 e−05	8.65 e−01	1.58 e−05
C6	1 e−05	2.0000001 e−25	$2.24e+12$	1.89 e−08	8.66 e−01	1.89 e−08
C10	5 e−01	1.9550825 e−02	7.08 e+00	2.10 e−01	3.65 e+01	3.13 e−01
C10	3 e−01	1.9690832 e−04	7.13 e+01	2.21 e−01	7.33 e+01	4.35 e−01
C10	1 e−01	1.0000004 e−08	1.00 e+04	3.33 e−01	1.04 e+02	5.95 e−01
C10	1 e−03	5.9991786 e−27	$1.29e+13$	5.27 e+00	1.17 e+01	4.99 e−01
C10	1 e−05	4.1102231 e−45	$1.56e+22$	7.69 e+01	2.63 e+00	2.70 e−01

Outline

Conclusions

Main points:

- asymptotic analysis of RVR as link reliabilities increase;
- with the balanced version, BRE is verified;
- a zero-variance IS approximation with BRE (and even VRE, in some cases) is described;
- the gain associated with the proposed technique is illustrated.
- A multinomial version can be used for a faster generation, together with polynomial structure reductions.