Error Analysis for Fourier Methods for Option Pricing

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Overview

- Introduction to European option pricing
- Numerical method and error control, 1D case
- Multi-D Basket Option Pricing
- Summary, references
Introduction to European option pricing

Numerical method and error control, 1D case

Multi-D Basket Option Pricing

Summary, references
European option: Derivative contract that gives the holder the right, but not the obligation, to buy or sell the underlying instrument at a specified price \( (K, \text{called strike}) \) at a future instant of time \((T, \text{called maturity})\).

Valuing an option: compute the fair price of this contract.
**Notation**


- **Asset:** \( \{ S_t : t \geq 0 \} \) random process such that \( S_t \in \mathbb{R}^n \) represents the price of the underlyings at time \( t \).

- **Maturity:** \( T \) is the maturity time of our contract.

- **Payoff:** \( G : \mathbb{R}^n \rightarrow \mathbb{R} \) determines that the holder receives the random amount \( G(S_T) \).

- **Interest rate:** Constant short rate \( r \).
To allow for sudden fluctuation of the markets, processes with jumps have become popular, expanding on more classical models such as the one of Black-Scholes.

We assume the logarithm of the asset prices to be a Lévy processes.

\[ S_t = S_0 e^{X_t} \quad \{X_t\}_{t \in [0,T]} \text{ a Lévy process} \]
Lévy processes

- Random process with independent stationary increments
- Characterized by the so called characteristic triplet \((\gamma, \sigma^2, \nu)\)
- The infinitesimal generator of a 1D Lévy process \(X\) is given by

\[
\mathcal{L}^X f(x) \equiv \lim_{h \to 0} \frac{\mathbb{E}(f(X_{t+h})|X_t = x) - f(x)}{h} = \gamma f'(x) + \frac{1}{2} \sigma^2 f''(x) + \int_{\mathbb{R}\setminus\{0\}} (f(x+y) - f(x) - y1_{|y|\leq 1}f'(x)) \nu(dy)
\]

- Fourier transform convention: \(\mathcal{F}[f](\omega) \equiv \int_{\mathbb{R}} e^{i\omega x} f(x) dx\)
- \(\mathcal{F}[\mathcal{L}^X f] = \Psi(-i\omega) \hat{f}(\omega)\), where \(\Psi(\cdot)\) is the characteristic exponent of the process \(X\) (defined by \(\mathbb{E}(e^{zX_t}) = e^{t\Psi(z)}\)).
In an arbitrage-free market there exists a measure, called risk-neutral measure, such that the price of a derivative is given by the expected value of the discounted future payoff.

Assume we work under a risk-neutral measure throughout the presentation. The price at time $t = T - \tau$ of a European option with payoff $G$ and maturity time $T$ is given by

$$\Pi(\tau, s) = e^{-r\tau} \mathbb{E}(G(S_T) | S_{T-\tau} = s)$$

where $\tau: 0 \leq \tau \leq T$ is the time to maturity.
The risk-neutral assumption on \((S_t)\), \(\mathbb{E} \left( e^{-r(h)S_{t+h}} \mid S_t = s \right) = s\), implies

\[
\int_{|y|>1} e^y \nu(dy) < \infty
\]

\[
\gamma = r - \frac{1}{2} \sigma^2 - \int_{\mathbb{R}\setminus\{0\}} (e^y - 1 - y 1_{|y|\leq 1}) \nu(dy)
\]

Expression for the characteristic Exponent \(\Psi(\cdot)\) is

\[
\Psi(z) = \left( r - \frac{\sigma^2}{2} \right) z + \frac{\sigma^2}{2} z^2 + \int_{\mathbb{R}} (e^{zy} - 1 - (e^y - 1)z) \nu(dy)
\]
Consider $g$ as the payoff function in log prices (i.e. defined by $g(x) = G(S_0 e^x)$). Now, take $f$ to be defined as

$$f(\tau, x) \equiv \mathbb{E}(g(X_T) | X_{T-\tau} = x)$$

Then $f$ solves the following PIDE:

$$\begin{cases} 
\partial_\tau f(\tau, x) = \mathcal{L}_x f(\tau, x) \\
\quad f(0, x) = g(x), \quad (\tau, x) \in [0, T] \times \mathbb{R}
\end{cases}$$
Taking Fourier transform with respect to $x$ we conclude that $\hat{f}$ is the solution of the ODE:

\[
\begin{aligned}
\frac{\partial_\tau \hat{f}(\tau,\omega)}{\hat{f}(\tau,\omega)} &= \Psi(-i\omega) \\
\hat{f}(0,\omega) &= \hat{g}(\omega)
\end{aligned}
\]  

(1)

Assume for the moment that $g$ is in $L^1$ and that $\hat{g}$ exists. Value function $f$ and price $\Pi$ are related through discounting

\[
\Pi(\tau, S_0 e^x) = e^{-r\tau} f(\tau, x)
\]  

(2)
Why to integrate in the Fourier space?

Questions:

- Is it worth complicating the approach by considering the problem in the Fourier space?
  - Explicit formula for the Characteristic Function
  - Fourier transform of the damped payoff is smooth

Problem: \( g \) typically not in \( L^1 \).

- 

\[
\int_{\mathbb{R}} p(y) g(y) \, dy = \int_{\mathbb{R}} (e^{\alpha y} p(y)) \left( g(y) e^{-\alpha y} \right) \, dy
\]

- truncating

\[
\int_{\mathbb{R}} p(y) g(y) \, dy \approx \int_{\mathbb{R}} p(y) g(y) 1_{|y| < y_{\text{max}}} \, dy
\]
Consider a damped version of $f$ defined by $f_\alpha(\tau, x) = e^{-\alpha x} f(\tau, x)$.

$\hat{f}_\alpha$ solves the ODE

$$
\begin{align*}
\frac{\partial_\tau \hat{f}_\alpha(\tau, \omega)}{\hat{f}_\alpha(\tau, \omega)} &= \Psi (\alpha - i \omega) \\
\hat{f}_\alpha(0, \omega) &= \hat{g}_\alpha(\omega)
\end{align*}
$$

(3)

where $\hat{g}_\alpha = \mathcal{F}[e^{-\alpha x} g(x)]$.

We get an explicit expression

$$
\hat{f}_\alpha(\tau, \omega) = e^{\tau \Psi (\alpha - i \omega)} \hat{g}_\alpha(\omega)
$$

This solution generally works for 1D problems.

For multi-dimensional problems the problem is more complicated. Truncating the payoff function ($g(x) \approx g(x) 1_{[a,b]^d}(x)$) allows handling the problem, but introduces an error.
inverse Fourier transform gives

\[ f_\alpha(\tau, x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega x} \hat{f}_\alpha(\tau, \omega) d\omega \]

\[ = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega x} e^{\tau \Psi(\alpha - i\omega)} \hat{g}_\alpha(\omega) d\omega \]

\[ \approx \frac{1}{2\pi} \int_{|\omega|<\omega_{\text{max}}} e^{-i\omega x} e^{\tau \Psi(\alpha - i\omega)} \hat{g}_\alpha(\omega) d\omega \]

- Once the previous formula is evaluated we undamp to recover \( f \)
- For the iFT integral, we use a trapezoidal quadrature
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Summary, references
Trapezoidal rule

Approximation $^1$ of $f_\alpha(\tau, x)$:

$$f_{\alpha, \Delta \omega, n}(\tau, x) = \frac{\Delta \omega}{2\pi} \sum_{k=-n}^{n-1} e^{-i(k+\frac{1}{2})\Delta \omega} \hat{f}_\alpha \left( \tau, \left( k + \frac{1}{2} \right) \Delta \omega \right)$$

$$= \frac{\Delta \omega}{\pi} \sum_{k=0}^{n-1} \text{Re} \left[ e^{-i(k+\frac{1}{2})\Delta \omega} \hat{f}_\alpha \left( \tau, \left( k + \frac{1}{2} \right) \Delta \omega \right) \right]$$

- The amount of computational work is determined by $n$
- How to choose $\alpha$ and $\Delta \omega$ to minimize the error for fixed $n$?

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$^1$This section is largely based on: Crocce, F., Häppölä, J., Kiessling, J., & Tempone, R. (2015). [Crocce, 2015]
Error bound

When approximating the integral by a finite sum we commit an error that can be split:

\[
\left| \int_{\mathbb{R}} - \sum_{n} \right| \leq \left| \int_{\mathbb{R}} - \sum_{\infty} \right| + \left| \sum_{\infty} - \sum_{n} \right| \leq \mathcal{E}
\]

(4)

- Under certain assumptions the quadrature error goes to zero at spectral rate
- The rate of convergence for the cutoff error depends on the tails of the distribution of the Lévy process and on the Fourier transform of the damped payoff.

Derive a bound similarly decomposed into quadrature and truncation parts:

\[
\mathcal{E} = \mathcal{E}_Q + \mathcal{E}_F
\]

(5)
Theorem (Quadrature error bound)

Assume that for $a > 0$:

H1. The characteristic function of $X_1$ has an analytic extension to the set

$$A_a^- \equiv \{ z \in \mathbb{C}: |\text{Im}[z] + \alpha| < a \}$$

H2. the Fourier transform of $g_\alpha(x)$ is analytic in the strip $A_a$ and

H3. there exists a continuous function $\gamma \in L^1(\mathbb{R})$ such that

$$|\hat{f}_\alpha(\tau, \omega + i\beta)| < \gamma(\omega) \text{ for all } \omega \in \mathbb{R} \text{ and for all } \beta \in [-a, a]$$

Then the quadrature error is bounded by

$$\mathcal{E}_Q \leq e^{\alpha x} \frac{M_{\alpha, a}(\tau, x)}{2\pi \left( e^{2\pi a/\Delta \omega} - 1 \right)} \equiv \bar{\mathcal{E}}_Q$$ (6)

with

$$M_{\alpha, a}(\tau, x) \equiv \sum_{\beta \in \{-a, a\}} \int_{\mathbb{R}} \left| e^{-i(\omega + i\beta)x} \hat{f}_\alpha(\tau, \omega + i\beta) \right| d\omega$$ (7)
Theorem (Simpler version)

Assume that: $\alpha$ and $a$ are such that

- $\int_{y>1} e^{(\alpha+a)y} \nu(dy) < \infty \quad \text{and} \quad \int_{y<-1} e^{(\alpha-a)y} \nu(dy) < \infty$
- $\hat{g}_\alpha \in L^\infty_{A_a}$
- $\sigma^2 > 0$ or there are enough small jumps to consider them as diffusive term ($C(\lambda) > 0$ for some $\lambda \in (0, 2)$)

Then the conclusion of the previous theorem holds with

$$\tilde{M}_{\alpha,a}(\tau, x) = \sum_{c \in \{-1, 1\}} e^{cax} e^{\tau \Psi(c\alpha)} |\hat{g}_\alpha(c\alpha)| \int_{\mathbb{R}} e^{-\tau \left( \frac{\sigma^2}{2} \omega^2 + \frac{|\omega|^{2-\lambda}}{4} C(\lambda) \mathbf{1}_{|\omega|>1} \right)} d\omega$$

$$C(\lambda) = \inf_{\kappa > 1} \left\{ \kappa^\lambda \int_{0<|y|<\frac{1}{\kappa}} y^2 \nu(dy) \right\}$$

In practice, $C(\lambda)$ can be evaluated analytically. If $\sigma^2 > 0$, we may approximate $C(\lambda) = 0$. 
Truncation error

The truncation error can be estimated similarly as

\[ \mathcal{E}_F \leq \int_{|\omega| > \omega_{\text{max}}} \left| \hat{f}_\alpha(\tau, \omega + i\beta) \right| \, d\omega \]  \hspace{1cm} (9)

In case \( C(\lambda) \) defined by eq. (8) is finite and positive for some \( \lambda \in (0, 2) \), this can be simplified as

\[ \overline{\mathcal{E}}_F \leq e^{\tau \Psi(0)} |\hat{g}_\alpha(0)| \int_{|\omega| > \omega_{\text{max}}} e^{-\tau \left( \frac{\sigma^2 \omega^2}{2} + \frac{|\omega|^2 - \lambda}{4} C(\lambda) \mathbf{1}_{|\omega| > 1} \right)} \, d\omega. \]  \hspace{1cm} (10)

Finally,

\[ \overline{\mathcal{E}} = \overline{\mathcal{E}}_F + \overline{\mathcal{E}}_Q. \]  \hspace{1cm} (11)
In order to satisfy a pre-determined error tolerance, we use the following iterative scheme:

1. Set $n = n_0$
2. Minimize bound (4)
3. If $\bar{E}$ does not meet tolerance, set $n \mapsto 2n$ and go to 2.

Bound magnitude depends exponentially on $x$. For call options in-the-money setting large damp $\alpha \gg 1$ is optimal. In deep out-of-the-money the most efficient solution is to exploit put-call-parity.
Figure: The true error $\mathcal{E}$ and the bound $\overline{\mathcal{E}}$ for the dissipative Merton model, for a range of quadrature points $n$, along with the bound-minimising configurations contrasted to the true error. $g(x) = 1_{[95,105]}$
In practice, the evaluation of eq (6) reduces to an $L^1$ norm for each component.

Integrals (9) and (2) are evaluated using with Clenshaw Curtis quadrature.

Evaluation of a relevant Variance Gamma model in milliseconds on a desktop computer, with little optimization.

To select appropriate parameters, we evaluate the bound $\mathcal{E}$ multiple times in a minimization scheme.

Proven existence (not uniqueness) of an optimizing configuration that minimizes $\mathcal{E}$.
Comparison to earlier work

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<thead>
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<th>$K$</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
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<td>$12\tau = 1$</td>
<td>$\alpha$</td>
<td>$-16.9$</td>
<td>$-13.8$</td>
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<td>$229$</td>
<td>$363$</td>
<td>$363$</td>
<td>$424$</td>
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<td></td>
<td>$\bar{E}$</td>
<td>$3.3 \times 10^{-4}$</td>
<td>$3 \times 10^{-3}$</td>
<td>$5 \times 10^{-3}$</td>
<td>$4 \times 10^{-4}$</td>
<td>$7 \times 10^{-6}$</td>
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<tr>
<td></td>
<td>$\bar{E}^*$</td>
<td>$6 \times 10^{-4}$</td>
<td>$3 \times 10^{-3}$</td>
<td>$5 \times 10^{-3}$</td>
<td>$6 \times 10^{-4}$</td>
<td>$1 \times 10^{-4}$</td>
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<td>$-13.8$</td>
<td>$22.1$</td>
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<tr>
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<td>$6.11$</td>
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<td></td>
<td>$\bar{E}$</td>
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<td>$3 \times 10^{-3}$</td>
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<td>$3 \times 10^{-4}$</td>
<td>$1 \times 10^{-5}$</td>
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<tr>
<td></td>
<td>$\bar{E}^*$</td>
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<td>$5 \times 10^{-3}$</td>
<td>$0.005$</td>
<td>$9 \times 10^{-4}$</td>
<td>$1 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

**Table:** The error bound for European call/put options in the VG model. Reference result $\bar{E}^*$ from [Lee, 2004]
Comparison to earlier work (2)

<table>
<thead>
<tr>
<th>$K$</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
</tr>
</thead>
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<td>$\bar{E}$</td>
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<td>$3.49 \times 10^{-4}$</td>
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<td>$6.77 \times 10^{-4}$</td>
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<tr>
<td>$\alpha$</td>
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<td>$-1.57$</td>
<td>$-1.57$</td>
<td>$-1.57$</td>
<td>$-1.57$</td>
</tr>
<tr>
<td>$\omega_{\text{max}}$</td>
<td>22.9</td>
<td>22.8</td>
<td>22.6</td>
<td>22.5</td>
<td>22.4</td>
</tr>
<tr>
<td>$\bar{E}^{*}$</td>
<td>0.34</td>
<td>0.26</td>
<td>0.21</td>
<td>0.17</td>
<td>0.13</td>
</tr>
<tr>
<td>$\bar{E}^{†}$</td>
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<td>$1.90 \times 10^{-3}$</td>
<td>$2.82 \times 10^{-3}$</td>
<td>$2.72 \times 10^{-3}$</td>
<td>$2.29 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

Table: Numerical performance of the bound for the Kou model, with the test case in with the number of quadrature points set to $n = 32$. 
**Figure**: The true error and the error bound for evaluating at the money options for the VG model test case. 

\[ \psi(z) = -\frac{1}{\tau} \log \left( 1 + i\nu z - \nu^2 \kappa^2 z^2 \right) \]
The model in question specifies a range \((z_+, z_-)\) in which characteristic function is defined.

Interval can be finite or infinite

For the case of call options \(g(x) = (e^x - e^k)^+\) the Fourier transform \(\hat{g}(z)\) has a pole at \(z = i\) and at \(z = 0\).

For put options, the \(\hat{g}\) equals that of call options.

For call options, the analyticity condition limits the parameter space

\[ a < |z_+ - \alpha| \]

\[ a < |\alpha - 1| \]
Parameter space
FFT allows us evaluation of the option price for multiple values $x$s simultaneously

$$\mathcal{E} \propto e^{\alpha x - a} + e^{\alpha x + a}$$

The Lévy model in question sets limits for $\alpha$

- if $\Psi(z)$ is defined for $|z| > 5$ (e.g. Merton model) considering $\alpha$ separately for each $x$ is often feasible.
- For other models, one may use same $\alpha$ independent of $x$ and exploit the computational gains of FFT

$$\Psi(z)_{\text{Merton}} = z \left( r - \frac{\sigma^2 z}{2} \right) + \frac{\sigma^2 z^2}{2} + \lambda \left( e^{\sigma^2 r \frac{z^2}{2}} - 1 - z \left( e^{\sigma^2 r \frac{z^2}{2}} - 1 \right) \right)$$
Figure: The true error $\mathcal{E}$ for the two VG test cases presented in Table 1 and the bound-minimising configurations (orange circle) $(\alpha_n, \Delta \omega_n)$ for the examples.
Introduction to European option pricing

Numerical method and error control, 1D case

Multi-D Basket Option Pricing

Summary, references
Multi-Dimensional Basket Call Option

Given $d$ underlyings $S \in \mathbb{R}^d$ with portfolio weights $c \in \mathbb{R}^d$ the call’s payoff is:

$$G(S_T) = (c'S_T - e^k)^+$$

The price is:

$$\Pi(s, \tau) = e^{-rT} \mathbb{E} (G(S_T)|S_{T-\tau} = s)$$

The complexity of the tensorized trapezoid quadrature grows exponentially (curse of dimensionality)

$$\mathbb{E} (G(S_T)|S_{T-\tau} = s) = \int_{\mathbb{R}} G(y)p(y)dx$$
Sparse Grid Approach

If the number of points of an univariate rule is \( n^1_l = O(2^l) \):

- Full tensor quadrature: \( n^d_l = O(2^{ld}) \)
- Smolyak’s sparse grid approach: \( n^d_l = O(2^l \cdot l^{d-1}) \).
- We want to integrate \((c' S_T - K)^+\), which is not smooth
Regularity through Conditional Expectation

\[
\mathbb{E} \left( \left( \sum_{i=1}^{d} c_i S_{iT} - K \right)^+ \right) = \mathbb{E} \left( c_1 S_{1T} - \left( K - \sum_{i=2}^{d} c_i S_{iT} \right) \right)^+ = \mathbb{E}_{S_{2T}, \ldots, S_{dT}} \left( \mathbb{E}_{S_{1T}} \left( c_1 S_{1T} - K' \mid S_{2T}, \ldots, S_{dT} \right) \right)
\]

\[
= \int_{\mathbb{R}^{d-1}} \left( \int (c_1 S_{1T} - K')^+ \cdot f \left( S_{1T} \mid S_{2T}, \ldots, S_{dT} \right) \, dS_{1T} \right) \cdot f(S_{2T}, \ldots, S_{dT}) \, dS_{2T}, \ldots, dS_{dT}.
\]
\[ \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}} (c_1 S_{1T} - K')^+ \cdot f(S_{1T} | S_{2T}, \ldots, S_{dT}) dS_{1T} \right) \cdot f(S_{2T}, \ldots, S_{dT}) dS_{2T}, \ldots, dS_{dT}. \]

- \( \int_{\mathbb{R}^d} \, dx \): a sparse grid integral
- \((d - 1)\)-dim pdf \( f \): from the characteristic function

- \( \int_{\mathbb{R}} \cdot \, dx \): comes from a 1D quadrature rule adapted to the discontinuity at \( X_{1T} = \log \left( \frac{K'}{c_1 S_{10}} \right) \)
- conditional pdf \( f \) from the cdf.
Example: the Multi-Dimensional Common Clock VG

- $B(t)$: d-dimensional Brownian motion with covariance rate $\Sigma$
- $G(t)$ univariate gamma process
- $\theta$: d-dimensional drift vector

Multivariate VG Process:

$$X(t) = \theta G(t) + B(G(t))$$

pdf:

$$f_{VG(t)}(x) = \frac{2e^{x'\Sigma^{-1}\theta}}{\nu \frac{t}{\nu} (2\pi)^{\frac{N}{2}} |\Sigma|^{\frac{1}{2}} \Gamma(\frac{t}{\nu})} \left( \frac{x'\Sigma^{-1}x}{\left( \frac{2}{\nu} + \theta'\Sigma^{-1}\theta \right)} \right)^{\frac{t}{2\nu} - \frac{N}{4}} K_{\frac{t}{\nu} - \frac{N}{2}} \left( \sqrt{(x'\Sigma^{-1}x)\left( \frac{2}{\nu} + \theta'\Sigma^{-1}\theta \right)} \right)$$

Characteristic Function:

$$\varphi_{CCVG}(u) = (1 - \nu u'\theta + \frac{1}{2} \nu u'\Sigma u)^{- \frac{1}{\nu}}$$
2-dimensional CCVG pdf with parameters:

- $\theta = [0.02; 0.03]$
- $\nu = 0.1$
- $\Sigma = \begin{bmatrix} 0.04 & 0.012 \\ 0.012 & 0.09 \end{bmatrix}$

rate of convergence of a 3-dim example:

- $\nu = 0.1$
- $\theta = [0.003; 0.002; 0.001]$
- $T = 1$
- $K = 120$
- $\Sigma = \begin{bmatrix} 0.04 & 0.006 & 0.009 \\ 0.006 & 0.09 & 0.018 \\ 0.009 & 0.018 & 0.09 \end{bmatrix}$
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Summary

- Derived an error analysis for Fourier methods in pricing European options
- Demonstrated improved performance of error bounds
  - Tightened bounds
  - General payoff function
- Automated evaluation of prices subject to error tolerance
- Preliminary work in multi-D framework


Thank You