Size Estimates in Inverse Problems

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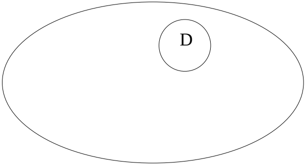


Ω electrical conductor

$D \subset \Omega$ inclusion

$$\gamma(x) = \begin{cases} 1 & x \in \Omega \setminus D \\ k & x \in D \end{cases}$$

Prescribed voltage $f \in H^{1/2}(\partial\Omega) \Longrightarrow$ potential u

$$\begin{cases} \mathrm{div}((1 + (k-1)\chi_D)\nabla u) = 0, & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

$\frac{\partial u}{\partial \nu}_{|\partial\Omega}$ current density we measure on the boundary

$\left\{ f, \frac{\partial u}{\partial \nu}_{|\partial\Omega} \right\}$ boundary measurements

Since uniqueness of the solution u, we may define

$$\begin{array}{rcccc} \Lambda_D & : & H^{1/2}(\partial\Omega) & \to & H^{-1/2}(\partial\Omega) \\ & & f & \to & \frac{\partial u}{\partial \nu}_{|\partial\Omega} \end{array}$$

This is called Dirichlet–to–Neumann map and its knowledge corresponds to performing infinitely many boundary measurements.

Inverse Problems: from $\Lambda_D \longrightarrow D$.

Uniqueness: Isakov 1988

- Fundamental solutions
- Runge Approximation Theorem

Stability

D_1, D_2 two inclusions such that their measurements are close

$$\|\Lambda_{D_1} - \Lambda_{D_2}\| \leq \varepsilon$$

Does this imply any information on the distance of the inclusions?

$$d(D_1, D_2) \leq \omega(\varepsilon)$$

Alessandrini-DC 2005

$$\omega(t) = C|\log t|^{-\eta}$$

- Fundamental Solution
- Quantitative Estimates of Unique Continuation

Optimality: DC-Rondi 2003

What if only a finite number of measurements are available?

Uniqueness: open problem

What parameters associated to D can be effectively evaluated from the available boundary measurements?

Friedman 1987: first approach

1997/98 Alessandrini, Kang, Rosset, Seo, Sheen showed that it is possible to estimate the size of the inclusion in the domain from the available boundary measurements

The idea is to compare the solution u with the solution u_0 of the same problem when $D = \emptyset$, that is

$$\begin{cases} \Delta u_0 = 0, & \text{in } \Omega, \\ u_0 = f, & \text{on } \partial\Omega \end{cases}$$

Clearly u and u_0 are minimizers respectively of the functionals

$$\mathcal{F}_D(v) = \int_\Omega (1 + (k-1)\chi_D)|\nabla v|^2 - 2\int_{\partial\Omega} vf,$$

$$\mathcal{F}_0(v) = \int_\Omega |\nabla v|^2 - 2\int_{\partial\Omega} vf,$$

for every $v \in H^1(\Omega)$.

Introducing the quantities

$$W = \int_{\partial\Omega} f \frac{\partial u}{\partial \nu} d\sigma, \qquad W_0 = \int_{\partial\Omega} f \frac{\partial u_0}{\partial \nu} d\sigma,$$

which can be computed from boundary measurements, we have

$$-W_0 = \mathcal{F}_0(u_0) \leq \mathcal{F}_0(u) = -W - 2\int_D |\nabla u|^2$$

$$-W = \mathcal{F}_D(u) \leq \mathcal{F}_D(u_0) = -W_0 - \int_D |\nabla u_0|^2,$$

that, exploiting the quadratic structure of the energy integrals, leads to

$$C_1 \int_D |\nabla u_0|^2 \leq |W_0 - W| \leq C_2 \int_D |\nabla u_0|^2$$

The previous inequality suggests that the power gap $W - W_0$ can be used as an indicator of the presence of the inclusion as long as we can evaluate quantitatively the two integrals appearing.

Upper bound: if $\mathrm{dist}(D, \partial\Omega) \geq d_0 > 0$, by interior regularity estimates we have

$$\sup_D |\nabla u_0|^2 \leq \mathrm{const.} W_0,$$

which leads to

$$\int_D |\nabla u_0|^2 \leq \mathrm{const.} W_0 |D| \qquad \Longrightarrow \qquad |D| \geq \mathrm{const.} \frac{W_0 - W}{W_0}$$

Lower bound: troublesome

Example: $n = 2$, $\Omega = B_1(0)$, $D = B_\rho(0)$, $0 < \rho < 1$

$$f = sin(j\theta) \qquad u_0 = \frac{1}{j}\sin(j\theta) \qquad \text{and} \qquad W_0 = \frac{\pi}{j}$$

In this case we have

$$\int_D |\nabla u_0|^2 = W_0 \left(\frac{|D|}{\pi}\right)^j$$

It is not possible to bound $\int_D |\nabla u_0|^2$ from below by any fixed power of $|D|$ but it suggests an estimate of the type

$$\int_D |\nabla u_0|^2 \geq \text{const.} W_0 \left(|D|\right)^p$$

for some $p > 1$ if we can a priori constrain the oscillation properties of the boundary data

Quantitative formalization: if the boundary data f does not oscillating too much, then the vanishing rate of $|\nabla u_0|^2$ at interior points cannot be too high.

Quantitative estimates of unique continuation

Three Spheres Inequality: for every $0 < r_1 < r_2 < r_3$ and for every $x_0 \in \Omega$ such that $B_{r_3}(x_0) \subset \Omega$

$$\int_{B_{r_2}(x_0)} |\nabla u_0|^2 \leq C \left(\int_{B_{r_1}(x_0)} |\nabla u_0|^2 \right)^{\delta} \left(\int_{B_{r_3}(x_0)} |\nabla u_0|^2 \right)^{1-\delta},$$

where $C > 0$ and $0 < \delta < 1$ depend on the a priori data and the radii

Doubling Inequality for every $0 < r \le \bar{r}$ and $x_0 \in \Omega_r$

$$\int_{B_{2r}(x_0)} |\nabla u_0|^2 \le K \int_{B_r(x_0)} |\nabla u_0|^2$$

A_p property for every $0 < r \le \bar{r}$ and $x_0 \in \Omega_r$

$$\left(\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |\nabla u_0|^2 \right) \left(\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |\nabla u_0|^{-\frac{2}{p-1}} \right)^{p-1} \le B$$

where $K > 0$, $B > 0$ and $p > 1$ depend on the a priori data and

$$F[f] = \frac{\|f\|_{H^{1/2}(\partial\Omega)}}{\|f\|_{L^2(\partial\Omega)}}$$

Putting everything together it it possible to prove the following bounds

$$C_1 \frac{W_0 - W}{W_0} \le |D| \le C_2 \left(\frac{W_0 - W}{W_0} \right)^{\frac{1}{p}}$$

where C_1, C_2 depend on the a priori data and only C_2 on $F[f]$.

With the additional hypothesis

$$|\{x \in D \, : \, \mathrm{dist}(x, \partial D) > h\}| \ge |D| \qquad \text{fat inclusion}$$

then the bounds can be refined as follows

$$C_1 \frac{W_0 - W}{W_0} \le |D| \le C_2 \frac{W_0 - W}{W_0}$$

where C_1, C_2 depend on the a priori data.

These estimates can be proved for more general conductivities.

$$\begin{cases} \mathrm{div}((A\chi_{\Omega\setminus D} + B\chi_D)\nabla u) = 0, & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

where A and B are $n \times n$ matrix with $L^\infty(\Omega)$ entries. Knowing some bounds on the jump, for instance, if we deal with a more conducting inclusion, we assume there exist $\eta > 0$, $0 < \delta < 1$ such that

$$\eta A \leq B - A, \qquad B \leq \delta A, \qquad \text{a.e. in } \Omega.$$

We have

$$\frac{1}{\delta - 1} C_1 \frac{W_0 - W}{W_0} \leq |D| \leq \left(\frac{\delta}{\eta}\right)^{1/p} C_2 \left(\frac{W_0 - W}{W_0}\right)^{\frac{1}{p}}$$

Stability: open problem

Let us consider now when we have an inclusion in a domain Ω that represents a linearly elastic, isotropic and homogeneous material

Prescribing a displacement field $f \in H^{1/2}(\partial\Omega, \mathbb{R}^n)$,

$$\begin{cases} \mathrm{div}((\mathbb{C} + (\mathbb{C}^D - \mathbb{C})\chi_D)\nabla u) = 0, & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

where $\mathbb{C}$ and $\mathbb{C}^D$ are the elastic tensors of the body and the inclusion respectively.

Stability (all measurements): Alessandrini, DC, Morassi, Rosset 2013

Defining the Dirichlet–to–Neumann map Λ_D as

$$\begin{array}{rclcl}\Lambda_D & : & H^{1/2}(\partial\Omega) & \longrightarrow & H^{-1/2}(\partial\Omega) \\ & & u_{|\partial\Omega} & \longrightarrow & (\mathbb{C}\nabla u)\nu_{|\partial\Omega},\end{array}$$

we consider two possible inclusions D_1, D_2. If

$$\|\Lambda_{D_1} - \Lambda_{D_2}\|_{\mathcal{L}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))} \leq \varepsilon,$$

then

$$d_H(\partial D_1, \partial D_2) \leq \omega(\varepsilon),$$

where ω is an increasing function on $[0, +\infty)$ satisfying

$$\omega(t) \leq C|\log t|^{-\eta}, \text{ for every } \quad 0 < t < 1,$$

where $C > 0$ and η, $0 < \eta \leq 1$, are constants only depending on the a-priori data.

i) We introduce the fundamental solutions Γ^{D_1}, Γ^{D_2} for the Lamé system in the full space when $D = D_1, D_2$ respectively.

ii) We show that $(\Gamma^{D_1} - \Gamma^{D_2})(y, w)$ can be dominated linearly by $\Lambda_{D_1} - \Lambda_{D_2}$ when y, w are outside of Ω.

iii) We propagate the smallness of $(\Gamma^{D_1} - \Gamma^{D_2})(y, w)$ as y, w are moved inside of Ω in the connected component $\mathcal{G}$ of $\mathbb{R}^3 \setminus (\overline{D_1 \cup D_2})$ which contains $\mathbb{R}^3 \setminus \overline{\Omega}$.

iv) We examine the asymptotics of $(\Gamma^{D_1} - \Gamma^{D_2})(y, w)$ as y, w approach to a point P of $\partial D_1 \setminus \overline{D_2}$ (or $\partial D_2 \setminus \overline{D_1}$).

v) We evaluate the distance between D_1 and D_2 by matching the smallness estimates of Step iii) with the blowup asymptotics of Step iv).

Finite number of measurements: size estimates (Alessandrini, Morassi, Rosset)

Main difficulty: quantitative estimates of unique continuation

Unique continuation for elliptic systems is, in general, not true

Alessandrini, Morassi, Rosset 01-02 proved a result of strong unique continuation

Ad hoc formulation of Korn–type inequality

$$\int_{B_r} |\nabla u - W_r|^2 + \frac{1}{R^2}|u - u_r - W_r x|^2 \leq C \left(\frac{R}{r}\right)^{4n-2} \int_{B_R} |\hat{\nabla} u|^2,$$

for every $u \in H^1(B_R, \mathbb{R}^n)$ and every r, $0 < r < R$, where

$$u_r = \frac{1}{|B_r|}\int_{B_r} u, \qquad W_r = \frac{1}{2|B_r|}\int_{B_r} (\nabla u - (\nabla u)^T)$$

Plate equation: determine an elastic inclusion $D \subset\subset \Omega$, Ω bounded domain in $\mathbb{R}^2$,

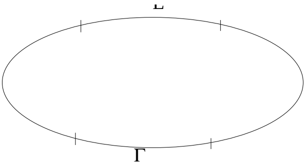
$$\mathrm{div}(\mathrm{div}((\chi_{\Omega\setminus D}\mathbb{P} + \chi_D\widetilde{\mathbb{P}})\nabla^2 w)) = 0, \quad \text{in } \Omega,$$

from one pair of Cauchy data $\{\widehat{M}_\tau, \widehat{M}_n\}$ and $\{w, w_{,n}\}$ s.t.

$$\begin{cases} M_n(w) := -(\mathbb{P}\nabla^2 w)n \cdot n = \widehat{M}_n, & \text{on } \partial\Omega, \\ V(w) := \mathrm{div}(\mathbb{P}\nabla^2 w) \cdot n + ((\mathbb{P}\nabla^2 w)n \cdot \tau)_{,s} = (\widehat{M}_\tau)_{,s}\,, & \text{on } \partial\Omega. \end{cases}$$

where $M_n(w)$ and $V(w)$ are respectively the *bending moment* and the *Kirchhoff shear* at $\partial\Omega$ associated to w. Here $\mathbb{P} = \frac{h^3}{12}\mathbb{C}$, with h the uniform thickness of the plate.

Boundary Corrosion: DC-Sincich-Vessella 2013



$\Gamma \subset \partial\Omega$ open portion where it is possible to perform measurements

$E \subset \partial\Omega$ not directly reachable where a corrosion process is going on

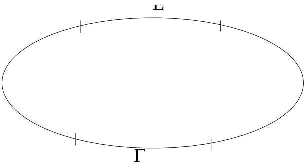
Surface Impedance

$$\gamma(x) = \gamma_0(x)\chi_\Gamma + k\chi_E, \qquad x \in \partial\Omega$$

where

$$k \text{ is an unknown constant,} \qquad \gamma_0 = 0 \text{ on } \partial\Omega \setminus \Gamma$$

Goal: estimate the size of E



g: prescribed current density such that

$$g = 0, \qquad \text{on } \partial\Omega \setminus \Gamma$$

We induce a potential u solution to the problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \gamma u = g, & \text{on } \partial\Omega, \end{cases}$$

where

$$\gamma(x) = \gamma_0(x)\chi_\Gamma + k\chi_E, \qquad x \in \partial\Omega$$

Inverse Problem: estimate the size of E from a knowledge of the Cauchy data $\{g, u_{|\Gamma}\}$

Main Theorem

$\Omega \subset \mathbb{R}^n$ bounded domain of class $C^{0,1}$
$\gamma, \gamma_0 \in L^\infty(\partial\Omega)$

There exist positive constants C_1, C_2, depending on the a priori data only, such that

$$C_1 \frac{W - W_0}{W_0} \leq |E| \leq C_2 \left(\frac{W - W_0}{W_0} \right)^{\frac{1}{p}}$$

where $p > 1$ is a constant depending on the a priori data only and $|\cdot|$ denotes the Lebesgue measure.