Multiscale Modeling of Wear Degradation in Cylinder Liners

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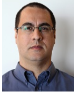

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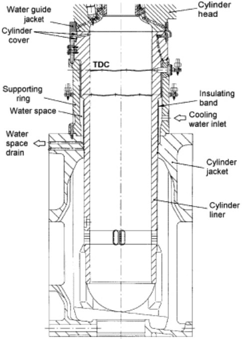
The problem

The data set consists of wear levels observed on 32 cylinder liners of eight-cylinder SULZER engines and measured by a caliper with a precision of $\Delta = 0.05$ mm. Warranty clauses specify that the liner should be substituted before it accumulates a wear level of 4.0 mm, in order to avoid expensive failures.



Question: when should we send the ship for maintenance?

The data

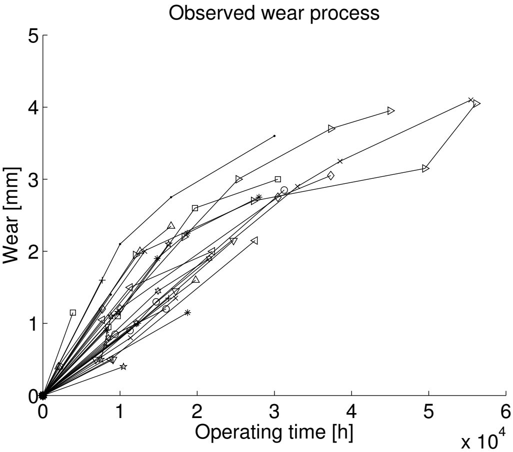


Figure : Due to the caliper's finite precision, every single measurement of the wear process, $W(t)$, belongs to the lattice $\{0, \Delta, 2\Delta, \ldots\}$.

Markov pure jump process

Modeling of Wear Degradation in Cylinder Liners, by A. Moraes, F. Ruggeri, R. Tempone and P. Vilanova. To appear in SIAM Multiscale Modeling and Simulation 2014.

- The evolution of the state vector, $X(t)$, is modelled as a continuous-time Markov chain (see [Norris, 1998]).
- Assume that each possible jump in the system occurs according to one of the pairs $\{(a_j(x;\theta), \nu_j)\}_{j=1}^{J}$, where $a_j : \mathcal{S}\times\Theta \to \mathbb{R}_+$ is known as the propensity function associated with the jump ν_j.
- The probability that the system jumps from $x\in\mathcal{S}$ to $x+\nu_j\in\mathcal{S}$ during the small interval $(t, t+dt)$ is

$$\mathrm{P}\left(X(t+dt) = x+\nu_j \;\middle|\; X(t) = x\right) = a_j(x;\theta)dt + o\left(dt\right).$$

The true likelihood I

For a continuously observed path of $X(s)$ in $[0, T]$, the likelihood function is:

$$L(\theta|X) := \exp\left(-\sum_{i=1}^{m+1} a_0(X(s_{i-1}),\theta)\,(s_i - s_{i-1})\right)\prod_{i=1}^{m} a_{j(s_{i-1})}(X(s_{i-1}),\theta),$$

where

- m is the total number of jumps in [0,T].
- $s_0=0$, $s_{m+1}=T$ and $\{s_1,\dots,s_m\}$ are the times in which $X(s)$ jumps.
- $j(s_i)$ is the index of the jump that occurs at time s_i.
- $a_0(x,\theta) := \sum_{j=1}^{J} a_j(x,\theta)$.

The true likelihood II

- Since we only observe X at times $\{0{=}t_0, t_1, \ldots, t_n{=}T\}$ we do not observe the jump times a.a. We need to maximize the expected likelihood,

$$\mathrm{E}\left[L(\theta|X)|\{X(t_0), \ldots, X(t_n)\}\right],$$

  for instance by a Monte Carlo version of the EM algorithm [Robert and Casella, 2005].
- Monte Carlo Sampling requires sampling from the exact distribution of the process X with parameter θ conditioned on $\{X(t_0), X(t_1), \ldots, X(t_n)\}$.
- Methods for sampling this kind of bridges are computationally intensive. Our multiscale methodology is focused on solving the inference problem by approximating the likelihood $L(\theta|X)$ by a simpler one.

Thickness process and inference goal

Let $X(t)$ be the *thickness process*, *i.e.*, $X(t) = X_0 - W(t)$, where W is the wear process, and X_0 is the initial thickness.
After a systematic model selection process, we obtained: $(a_1(x), \nu_1) = (c_1 x, -\Delta)$ and $(a_2(x), \nu_2) = (c_2 x, -k\Delta)$, where k is a positive integer to be determined.
The probability of observing a thickness decrease in a small time interval $(t, t + dt)$ is

$$P(X(t + dt) = X(t) - \Delta \mid X(t) = x) = c_1 x \, dt$$
$$P(X(t + dt) = X(t) - k\Delta \mid X(t) = x) = c_2 x \, dt$$

where $X(0) = x_0$ is the initial thickness and $\theta = (c_1, c_2, X_0, k)$ is the vector of unknown parameters.

Dynkin formula

Let X be a time-homogenous Markov process. Let us define the strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ of operators defined on a closed subspace in the class of bounded Borel measurable functions in a given metric space [Ethier and Kurtz, 2005] by

$$T(s)f(X(t)) := \mathrm{E}\left[f(X(t+s)) \,\middle|\, \mathcal{F}_t^X\right],$$

where $\mathcal{F}_t^X$ is the filtration generated by X. The infinitesimal generator, $\mathcal{L}_X$, of the semigroup $\{T(t)\}_{t\geq 0}$ is defined by

$$\mathcal{L}_X(f) := \lim_{t\to 0} \frac{1}{t}\{T(t)f - f\},$$

whenever this limit exists. The Dynkin formula [Klebaner, 2005] states the following identity:

$$\mathrm{E}\left[f(X(t))\right] = f(X(0)) + \int_0^t \mathrm{E}\left[\mathcal{L}_X(f)(s)\right] ds.$$

Upscaling via infinitesimal generators

The generator $\mathcal{L}_X$ of a pure jump Markov process X

$$\mathcal{L}_X(f)(x,t) = \sum_j a_j(x;\theta)(f(x+\nu_j(\theta),t) - f(x,t)).$$

A formal Taylor expansion of order one gives

$$\mathcal{L}_Z(f)(x,t) = \sum_j a_j(x;\theta)\partial_x f(x,t)\nu_j(\theta),$$

which corresponds to reaction-rates ODE (mean field)

$$\begin{cases} dZ(t) & = & \nu(\theta)a(Z(t);\theta)dt, \;\; t \in \mathbb{R}_+ \\ Z(0) & = & x_0 \in \mathbb{R}_+. \end{cases} \tag{1}$$

where the $j-$column of the matrix ν is $\nu_j(\theta)$, and a is a column vector with components a_j.

Mean Field model

The data $\mathbf{x} = \{x_i\}_{i=1}^n$ are modeled according to

$$x_i = Z(t_i) + \epsilon_i \text{ (model 1)}$$

where: $Z(t)$ satisfies the mean field ODE, ϵ_i are i.i.d. realizations of $\mathcal{N}(0, \Delta^2)$ for $i = 1, \ldots, n$.
In this case, the likelihood can be written as

$$L(\theta; \mathbf{x}) \propto \prod_{i=1}^{n} \exp\left\{-\frac{(x_i - Z(t_i; \theta))^2}{2\Delta^2}\right\},$$

where $\theta=(c_1, c_2, X_0, k)$.
Now, the maximum likelihood estimator (MLE) for θ, is given by the minimizer of minus the log likelihood,

$$\theta^* = \arg\min_{\theta\in\Theta} \sum_{i=1}^{n} \left\{(x_i - Z(t_i; \theta))^2)\right\}.$$

A Gaussian moment expansion model I

The data $\mathbf{x} = \{x_i\}_{i=1}^{n}$ are modeled according to

$$x_i = Z(t_i) + \epsilon_i \ \textbf{(model 2)}.$$

Here $Z(t) \sim \mathcal{N}(m(t), \sigma^2(t))$, $m(t)$ and $\sigma^2(t)$ satisfy

$$\begin{cases} dm(t) & = & (c_1\nu_1 + c_2\nu_2)m(t)dt, \\ d\sigma^2(t) & = & (2(c_1\nu_1 + c_2\nu_2)\sigma^2(t) + (c_1\nu_1^2 + c_2{\nu_2}^2)m(t))dt, \\ (m(0), \sigma^2(0)) & = & (x_0, 0),\ x_0 \in \mathbb{R}_+,\ t \in \mathbb{R}_+, \end{cases}$$

and ϵ_i are i.i.d. realizations of $\mathcal{N}(0, \Delta^2)$ for $i = 1, \ldots, n$.
In this case, the likelihood can be written as

$$L(\theta; \mathbf{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi(\Delta^2 + \sigma^2(t_i; \theta))}} \exp\left\{ -\frac{(x_i - m(t_i; \theta))^2}{2(\Delta^2 + \sigma^2(t_i; \theta))} \right\}.$$

A Gaussian moment expansion model II

The MLE for θ is given by the minimizer of minus the log likelihood,

$$\theta^* = \arg\min_{\theta \in \Theta} \sum_{i=1}^{n} \left\{ \frac{(x_i - m(t_i;\theta))^2}{\Delta^2 + \sigma^2(t_i;\theta)} + \log(\Delta^2 + \sigma^2(t_i;\theta)) \right\}.$$

We determine first the minimum conditioned on k and X_0 and then the global optimizer.

The Master Equation (ME)

The time dependent density of X, $p_x(t;\theta)$, satisfies a system of ODEs called the Master Equation ([Gardiner, 2010, Risken and Frank, 1996]). In our pure jump case is

$$\begin{cases} \frac{dp_x}{dt}(t;\theta) & = & \sum_j (p_{x+\nu_j}(t;\theta)a_j(x+\nu_j;\theta) \\ & - & p_x(t;\theta)a_j(x;\theta)), \;\; t \in \mathbb{R}_+, x, x+\nu_j \in \mathcal{S} \\ p_x(0;\theta) & = & \mathbf{1}_{\{x=x_0\}}, \end{cases}$$

For this simple problem, once θ is determined, this system of ODEs can be efficiently solved by standard numerical techniques.

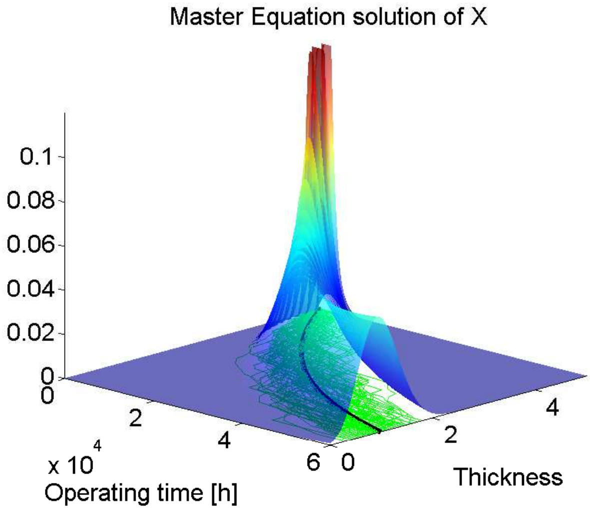


Figure : PMF solution of the Master Equation and 100 SSA paths.

Results I

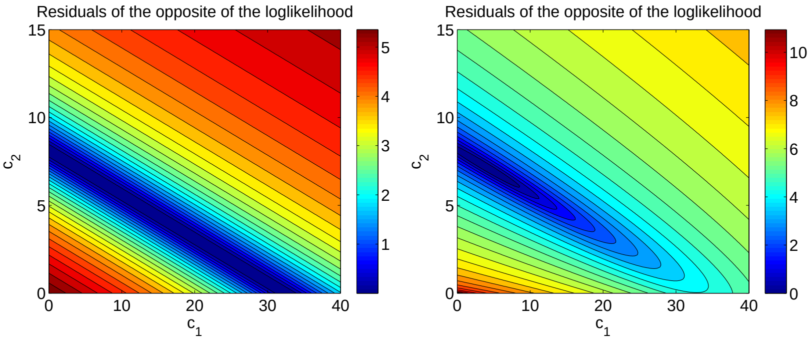


Figure : Residuals of the opposite of the loglikelihood for $k = 4$ and $X_0 = 5\,mm$. **Left panel**: Model 1. There is an identifiability problem for the parameters c_1 and c_2. **Right panel**: Model 2. Unique global maximum $(c_1^*, c_2^*) = (0.63 \cdot 10^{-4}, 1.2 \cdot 10^{-4})$.

Results II

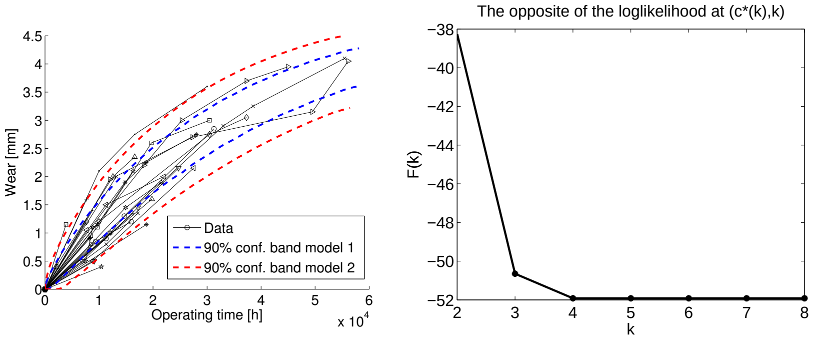


Figure : **Left panel**: the exact 90% confidence band computed from the Master Equation. **Right panel**: Plot the opposite of the likelihood as function of k for X_0=5 mm fixed.

Answer to the motivational question:

The ship should be sent to maintenance at the time that the thickness process, X, is less than or equal than $B = X_0 - 4mm$, where $X_0 = 5mm$ is the initial thickness.
We have that $F_{\tau_B;\theta}(t) := \mathrm{P}\left(X(t) \leq B|\theta\right) = \sum_{x \leq B} p_x(t;\theta)$, where $p_x(t;\theta)$ is the probability that $X(t){=}x$, given the value of the parameter vector θ.

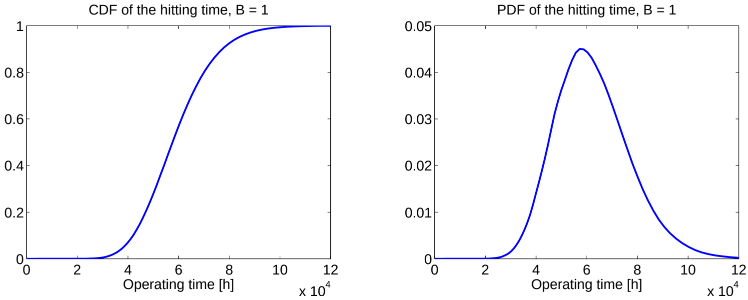


Figure : **Left panel**: CDF of the hitting time for $B = 1$. **Right panel**: PDF of the hitting time to the critical level.

Conditional Residual Reliability I

Suppose that we know that the wear process, W, is at level w_0 at time $t_0 \geq 0$. Assume that there exists a critical stopping level, $w_{max} > w_0$, that determines the residual lifetime $\tau_{max} - t_0$. For $t > 0$, the residual lifetime is greater than t, if and only if $W(t_0 + t) < w_{max}$. Therefore, the conditional probability

$$\mathrm{P}\left(\tau_{max} - t_0 > t | W(t_0) = w_0\right) = \mathrm{P}\left(W(t_0 + t) < w_{max} | W(t_0) = w_0\right).$$

Taking into account the relation between the wear and the thickness processes, we have that the conditional residual reliability function defined as

$$R(t; t_0, w_0) := \mathrm{P}\left(\tau_{max} - t_0 > t | W(t_0) = w_0\right)$$

can be written as $\mathrm{P}\left(X(t; X_0 - w_0) > X_0 - w_{max}\right)$, where $X(\cdot, x_0)$ is the thickness process starting from x_0.

Conditional Residual Reliability II

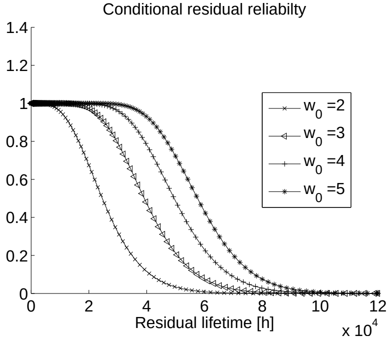


Figure : Behavior of the conditional residual reliability function, $R(t;0,w_0)$ for some values of w_0. In this case, we set $w_{max} = 4$. As expected, for a fixed residual lifetime t, we have that $R(t;0,w_0)$ is a decreasing function of w_0.

Conclusions

- We presented a novel approach to the problem of modeling the wear process of cylinder liners [Moraes et al., 2014].
- We found that the wear process can be modeled using only two jumps of amplitudes Δ and 4Δ, with linear propensity functions. In contrast to the work of Giorgio et al (2011) [Giorgio et al., 2011], we did not need to use age-dependent propensity functions or gamma noise. Nevertheless, our approach is totally suitable for dealing with age-dependent propensities.
- One of the main contributions of this work is the multiscale indirect inference approach, where the inferences are based on upscaled models.
- Thanks to the remarkable simplicity of our model, we can easily obtain the distribution of any observable of the process directly from the solution of the Master Equation, which provides the probability distribution of the process at all times.

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