Long time behavior of finite state (potential) mean-field games via $\Gamma$-convergence

Rita Ferreira

Pos-Doc @ Instituto Superior Técnico

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Centro de Matemática e Aplicações (FCT-UNL)

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Abstract

In this talk we address the study of the long time behavior of continuous time, finite state (potential) mean-field games using $\Gamma$-convergence.

The method we propose allows extending existent results in literature as it requires weaker hypotheses than the ones usually assumed.

In collaboration with Diogo Gomes (KAUST)
Problem set-up

- very large number of indistinguishable players
- players can be in a finite number of states, \( I_d = \{1, 2, 3, \ldots, d\} \)
- distribution of the players among the different states is given by a probability vector \( \theta \in \mathcal{P}(I_d) \), where
  \[
  \begin{align*}
  \theta^1 + \ldots + \theta^d &= 1, \\
  \theta^i &\geq 0 \quad \forall i \in I_d.
  \end{align*}
  \]
- each player only knows its state and the probability \( \theta \)
- states evolve randomly in time by following a controlled continuous time Markov chain and each player controls its switching rate in order to optimize a certain functional:

  running cost \( c \) + terminal cost \( \psi \)
Problem set-up

Running cost:

\[ c : I_d \times \mathcal{P}(I_d) \times (\mathbb{R}^+_0)^d \rightarrow \mathbb{R} \]

\[ (i, \theta, \alpha) \mapsto c(i, \theta, \alpha) \]

- \( i \) ... state of the player
- \( \theta \) ... probability distribution of the population among states
- \( \alpha_j \) ... transition rate the player uses to change from state \( i \) to state \( j \)

Terminal Cost:

\[ \psi : I_d \rightarrow \mathbb{R} \]
• Assume $\alpha \mapsto c(i, \theta, \alpha)$ convex, not depending on $\alpha_i$

• Let $\Delta_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$ denote the difference operator with respect to $i$:

$$\Delta_i z := (z^1 - z^i, \ldots, z^d - z^i), \quad \text{for all } z = (z^1, \ldots, z^d) \in \mathbb{R}^d.$$ 

• We define the generalized Legendre transform of the function $c(i, \theta, \cdot)$ as

$$h(z, \theta, i) := \min_{\mu \in (\mathbb{R}_0^+)^d} \left\{ c(i, \theta, \mu) + \sum_{j=1}^d \mu_j (z^j - z^i) \right\}$$

$$= \min_{\mu \in (\mathbb{R}_0^+)^d} \left\{ c(i, \theta, \mu) + \mu \cdot \Delta_i z \right\},$$

which defines a concave function in $z$. 
Mean-Field Games

The value function is the unique solution of Hamilton-Jacobi ODE

\[-\dot{v}^i = h(\Delta_i v, \theta, i),\]

satisfying \(v(T) = \psi\).

The optimal switching policy for a player in state \(i\) is

\[\alpha^*_i(\Delta_j v, \theta, j) = \frac{\partial}{\partial z_j} (h(\Delta_i v, \theta, i)).\]

The mean-field equilibrium arises when all players use the same optimal switching policy, which gives rise to the system:
\[
\begin{align*}
\dot{\theta}^i &= \sum_{j=1}^{d} \theta^j \alpha^*_i(\Delta_j v, \theta, j), \\
-\dot{v}^i &= h(\Delta_i v, \theta, i),
\end{align*}
\]

together with the initial-terminal conditions

\[
\theta(0) = \theta_0, \quad v(T) = \psi,
\]

where \( \theta_0 \) is the initial distribution of players.

- \( v \) is the value function and \( \theta \) is the probability distribution of the population among states;
- \( h \) is known (it is related to the running cost function \( c \)) and is concave in the first variable;
- \( \alpha^*_i \) is known (provides optimal switching policy for a player in state \( i \));
- \( \Delta_i z := (z^1 - z^i, \ldots, z^d - z^i) \) for \( z = (z^1, \ldots, z^d) \in \mathbb{R}^d \).
Remark

From the ODE viewpoint these equations are non-standard as some of the variables have initial conditions whereas others have prescribed terminal data.

Potential Mean-Field Games

Corresponds to the case in which $h$ has the form

$$h(z, \theta, i) = \tilde{h}(z, i) + f(\theta, i),$$

where \( \frac{\partial F}{\partial \theta_i}(\cdot) = f(\cdot, i) \) for some convex function $F : \mathbb{R}^d \to \mathbb{R}$.

If $F$ strictly convex, superlinear growth at infinity, and non-increasing in each coordinate, then $(\nu, \vartheta)$ is a solution of the MFG iff

1. $\nu$ is a critical point of the functional

$$\int_0^T F^* (\dot{\nu} + \tilde{h}(\Delta \nu, \cdot)) \, dt - \theta_0 \cdot \nu(0),$$

where we are looking for critical points $\nu$ that have fixed boundary condition at $T$, namely $\nu(T) = \psi$

2. $\vartheta(t) = -\nabla F^*(\dot{\nu}(t) + \tilde{h}(\Delta \nu(t), \cdot))$. 
The stationary setting

Goal:
Study of long time convergence (trend to equilibrium problem) for finite state mean-field games

A triplet $(\bar{\theta}, \bar{v}, \bar{\lambda}) \in \mathcal{P}(I_d) \times \mathbb{R}^d \times \mathbb{R}$ is called a stationary solution of the mean-field equations if

$$\begin{align*}
\sum_{j=1}^{d} \bar{\theta}^j \alpha_i^*(\Delta j \bar{v}, \bar{\theta}, j) &= 0, \\
h(\Delta_i \bar{v}, \bar{\theta}, i) &= \bar{\lambda},
\end{align*}$$

for all $i \in I_d$.

If $(\bar{\theta}, \bar{v}, \bar{\lambda})$ is a stationary solution for the MFG equations, then $(\bar{\theta}, \bar{v} - \bar{\lambda} t \mathbf{1})$, where $\mathbf{1} := (1, \ldots, 1)$, solves the MFG equations.
Consider

\[
\min \left\{ \int_0^1 F^*(\tilde{h}(\Delta v, \cdot) - \lambda \mathbf{1}) \, dt - \lambda : \quad v : [0, 1] \rightarrow \mathbb{R}^d \text{ continuous}, \sum_{i=1}^d v_i(t) = 0, \lambda \in \mathbb{R} \right\}.
\]

Jensen’s inequality + $F^*$ convex, componentwise non-increasing function + $\tilde{h}(\cdot, i)$ concave function \Rightarrow it suffices to consider minimizers to this problem in the class of constant functions $v$. Therefore it is enough to look at minimizers of

\[
\min \left\{ F^*(\tilde{h}(\Delta v, \cdot) - \lambda \mathbf{1}) - \lambda : \quad v \in \mathbb{R}^d, \sum_{i=1}^d v^i = 0, \lambda \in \mathbb{R} \right\}.
\]

In turn, if $(\bar{v}, \bar{\lambda})$ is a critical point for the latter, then setting

\[
\bar{\theta}^j := -\frac{\partial F^*}{\partial p_j}(\tilde{h}(\Delta \bar{v}, \cdot) - \lambda \mathbf{1}), \quad j \in I_d,
\]

we conclude that $(\bar{\theta}, \bar{v}, \bar{\lambda})$ is a stationary solution of the mean-field eqns.
An important estimate

In $\mathbb{R}^d / \mathbb{R}$ we define the norm

$$ \|z\|_\# = \frac{\max_{i \in I_d} z^i - \min_{i \in I_d} z^i}{2}, \quad z \in \mathbb{R}^d. $$

Definition

Let $\langle v \rangle := \frac{1}{d} \sum_{j=1}^d v^j$. We say that $h$ is contractive if there exists $M > 0$ such that if $\|v\|_\# > M$, then the two following conditions hold for all $\theta \in \mathcal{P}(I_d)$ and $i \in I_d$:

- $(\Delta_i v)^j \leq 0$ for all $j \in I_d$ implies $h(\Delta_i v, \theta, i) - \langle h(v, \theta, \cdot) \rangle < 0$,
- $(\Delta_i v)^j \geq 0$ for all $j \in I_d$ implies $h(\Delta_i v, \theta, i) - \langle h(v, \theta, \cdot) \rangle > 0$.

Many mean-field games are contractive [GMS2013].

Moreover, for contractive, potential mean-field games:

If $\|v(T)\|_\# \leq M$, where $v$ is a solution, and $M$ is large enough, then $\|v(t)\|_\# \leq M$ for all $t \in [0, T]$. 
Scaling - study of long time behavior of mean-field games

We introduce a scaled version of the mean-field game, where $\epsilon = \frac{1}{T}$,

$$
\begin{align*}
\epsilon \dot{\theta}_\epsilon^i &= \sum_{j=1}^d \theta^i_j \alpha^*_i (\Delta_j v_\epsilon, \theta_\epsilon, j), \\
-\epsilon \dot{v}_\epsilon^i &= h(\Delta_i v_\epsilon, \theta_\epsilon, i),
\end{align*}
$$

together with the initial-terminal conditions

$$
\theta_\epsilon(0) = \theta_0, \quad v_\epsilon^i(1) = \psi^i.
$$

Wlog, $\sum_{i=1}^d \psi^i = 0$.

**Important remark:**

Scaling in time does not change the bounds mentioned above. Hence, for contractive, potential mean-field games:

$$
\sup_{t \in [0,1]} \sup_{\epsilon > 0} \|v_\epsilon(t)\|_\# < +\infty.
$$
In order to rewrite the associated functional in a convenient form for the use of $\Gamma$-convergence we decompose $v_\epsilon$ as follows.

Let $\lambda_\epsilon \in \mathbb{R}$, $u_\epsilon : [0, 1] \to \mathbb{R}^d$, and $w_\epsilon : [0, 1] \to \mathbb{R}$ be defined by

\[
\lambda_\epsilon := \int_0^1 \sum_{i=1}^d h(\Delta_i v_\epsilon, \theta_\epsilon, i) \, dt,
\]

\[
w_\epsilon(t) := \frac{\epsilon}{d} \sum_{i=1}^d v_i^\epsilon(t) - \lambda_\epsilon(1 - t),
\]

\[
u_i^\epsilon(t) := v_i^\epsilon(t) - \frac{1}{\epsilon} w_\epsilon(t) - \frac{\lambda_\epsilon}{\epsilon} (1 - t).
\]
Observing that $\Delta_i u_\epsilon = \Delta_i v_\epsilon$ for all $i \in I_d$, the (scaled) mean-field equations become

$$
\begin{align*}
\epsilon \dot{\theta}_\epsilon^i &= \sum_{j=1}^{d} \theta_j \alpha_i^*(\Delta_j u_\epsilon, \theta_\epsilon, j), \\
\lambda_\epsilon - \dot{w}_\epsilon - \epsilon \dot{u}_\epsilon^i &= h(\Delta_i u_\epsilon, \theta_\epsilon, i).
\end{align*}
$$

Moreover, for all $t \in [0, 1]$, $\epsilon > 0$, and $i \in I_d$,

$$
\begin{align*}
\sup_{\epsilon > 0} |\lambda_\epsilon| < +\infty, & \quad \sup_{t \in [0, 1]} \sup_{\epsilon > 0} \|u_\epsilon(t)\|_\# < +\infty, \\
\sum_{i=1}^{d} \epsilon \dot{u}_\epsilon^i(t) = 0, & \quad \epsilon \dot{u}_\epsilon^i(1) = \psi^i, \quad \sup_{\epsilon > 0} \|\epsilon \dot{u}_\epsilon\|_\infty < +\infty, \\
w_\epsilon(0) = 0, & \quad w_\epsilon(1) = 0, \quad \sup_{\epsilon > 0} \|\dot{w}_\epsilon\|_\infty < +\infty.
\end{align*}
$$
From the variational point of view, we look for minimizers of

\[ \int_0^1 F^*(\dot{w} \mathbf{1} + \epsilon \dot{u} + \tilde{h}(\nabla \cdot u, \cdot) - \lambda \mathbf{1}) \, dt - \epsilon \theta_0 \cdot u(0) - \lambda \]

over \( \lambda \in \mathbb{R} \), \( u : [0, 1] \to \mathbb{R}^d \), and \( w : [0, 1] \to \mathbb{R} \) according to the conditions above.

Expect to obtain in the limit:

\[ \int_0^1 F^*(\dot{w} \mathbf{1} + \tilde{h}(\nabla \cdot u, \cdot) - \lambda \mathbf{1}) \, dt - \lambda, \]

which corresponds to the *stationary functional* introduced before provided that \( w \) does not depend on \( t \).
A $\Gamma$-convergence result

- $X$ be a reflexive Banach space endowed with its weak topology
- $\mathcal{F}_n : X \to \overline{\mathbb{R}}$ equi-coercive in the weak topology of $X$
- exists $\mathcal{F} : X \to \overline{\mathbb{R}}$ such that
  - for every $x \in X$ and for every sequence $\{x_n\}_{n \in \mathbb{N}}$ weakly converging to $x$ in $X$, one has $\mathcal{F}(x) \leq \liminf_{n \to +\infty} \mathcal{F}_n(x_n)$;
  - or every $x \in X$ there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ weakly converging to $x$ in $X$ such that $\mathcal{F}(x) = \lim_{n \to +\infty} \mathcal{F}_n(x_n)$.

Then

- $\min_{x \in X} \mathcal{F}(x) = \lim_{n \to +\infty} \inf_{x \in X} \mathcal{F}_n(x)$;
- if $x_n$ is a $\delta_n$-minimizer of $\mathcal{F}_n$ in $X$, where $\delta_n \downarrow 0^+$, and $x$ is a cluster point of $\{x_n\}_{n \in \mathbb{N}}$, then $x$ is a minimizer of $\mathcal{F}$ in $X$, and $\mathcal{F}(x) = \limsup_{n \to +\infty} \mathcal{F}_n(x_n)$;
- if $\{x_n\}_{n \in \mathbb{N}}$ weakly converges to $x$ in $X$, then $x$ is a minimizer of $\mathcal{F}$ in $X$, and $\mathcal{F}(x) = \lim_{n \to +\infty} \mathcal{F}_n(x_n)$. 
Convergence of functionals, and its minima, associated with mean-field games

The space of continuous functions is not the most appropriate one for this study as it is not a reflexive Banach space.

We then extend the functionals to the product space $L^p \times W_0^{1,p}$, for $p \in (1, +\infty)$, in a natural way.

Assumptions

- $F^* : \mathbb{R}^d \rightarrow \mathbb{R}$ is componentwise non-increasing and convex
- $\tilde{h} : \mathbb{R}^d \times I_d \rightarrow \mathbb{R}$ is concave in the first variable
- $\hat{h} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the function defined by

  $$\hat{h}(z) := (\hat{h}^1(z), \cdots, \hat{h}^d(z)), \quad \text{with } \hat{h}^i(z) := \tilde{h}(\Delta_i z, i) \text{ for all } i \in I_d,$$
\[ F_\varepsilon : L^p((0, 1); \mathbb{R}^d) \times W^{1,p}_0(0, 1) \times \mathbb{R} \to \mathbb{R}, \]

\[ F_\varepsilon(u, w, \lambda) := \begin{cases} 
G_\varepsilon(u, w, \lambda) & \text{if } (u, w, \lambda) \in \Phi_\varepsilon, \\
+\infty & \text{otherwise}, 
\end{cases} \]

\[ G_\varepsilon(u, w, \lambda) := \int_0^1 F^*(\dot{w}(t)1 + \varepsilon\dot{u}(t) + \tilde{h}(u(t)) - \lambda 1) \, dt - \varepsilon \theta_0 \cdot u(0) - \lambda, \]

\[ \Phi_\varepsilon := \left\{ (u, w, \lambda) \in W^{1,p}((0, 1); \mathbb{R}^d) \times W^{1,p}_0(0, 1) \times \mathbb{R} : \right. \]

\[ u(1) = \psi, \|u(\cdot)\|_\# \leq \bar{M}_0 \text{ and } \sum_{i=1}^d u^i(\cdot) = 0 \mathcal{L}^1\text{-a.e. in } (0, 1), \]

\[ \max \left\{ \int_0^1 |\varepsilon\dot{u}(t)|^p \, dt, \|\dot{w}\|_{L^\infty(0,1)} \right\} \leq M_0, \, |\lambda| \leq R_0 \}

(some \( M_0, \bar{M}_0, R_0 \in \mathbb{R} \).)
\( \mathcal{F}_0 : L^p((0, 1); \mathbb{R}^d) \times W^{1,p}_0(0, 1) \times \mathbb{R} \to \overline{\mathbb{R}}, \)

\[
\mathcal{F}_0(u, w, \lambda) := \begin{cases} 
\mathcal{G}_0(u, w, \lambda) & \text{if } (u, w, \lambda) \in \Phi_0, \\
+\infty & \text{otherwise}, \end{cases}
\]

\[
\mathcal{G}_0(u, w, \lambda) := \int_0^1 F^*(\dot{w}(t)1 + \tilde{h}(u(t)) - \lambda 1) \, dt - \lambda,
\]

\[
\Phi_0 := \left\{ (u, w, \lambda) \in L^p((0, 1); \mathbb{R}^d) \times W^{1,p}_0(0, 1) \times \mathbb{R} : \right. \]

\[
\left. \left\| u(\cdot) \right\|_\# \leq \bar{M}_0 \text{ and } \sum_{i=1}^d u^i(\cdot) = 0 \mathcal{L}^1\text{-a.e. in } (0, 1), \right.
\]

\[
\left. \left\| \dot{w} \right\|_{L^\infty(0,1)} \leq M_0, \left| \lambda \right| \leq R_0 \right\}.
\]
Main Theorem

- \{F_\varepsilon\}_{\varepsilon>0} \Gamma\text{-converges as } \varepsilon \to 0^+ \text{ to } F_0 \text{ with respect to the weak convergence in } L^p((0, 1); \mathbb{R}^d) \times W_0^{1,p}(0, 1) \times \mathbb{R}.

- Let \((u_\varepsilon, w_\varepsilon, \lambda_\varepsilon)\) be a \(\delta_\varepsilon\)-minimizer, where \(\delta_\varepsilon \downarrow 0^+\), of \(G_\varepsilon\) in \(\Phi_\varepsilon\). Then \(\{(u_\varepsilon, w_\varepsilon, \lambda_\varepsilon)\}_{\varepsilon>0}\) is bounded in norm; if \((u, w, \lambda)\) is a cluster point, then \((u, w, \lambda)\) is a minimizer of \(G_0\) in \(\Phi_0\) and

\[
G_0(u, w, \lambda) = \limsup_{\varepsilon \to 0^+} G_\varepsilon(u_\varepsilon, w_\varepsilon, \lambda_\varepsilon).
\]

- If \(\{(u_\varepsilon, w_\varepsilon, \lambda_\varepsilon)\}_{\varepsilon>0}\) weakly converges to \((u, w, \lambda)\) in \(L^p((0, 1); \mathbb{R}^d) \times W_0^{1,p}(0, 1) \times \mathbb{R}\), then \((u, w, \lambda)\) is a minimizer of \(G_0\) in \(\Phi_0\), and

\[
G_0(u, w, \lambda) = \lim_{\varepsilon \to 0^+} G_\varepsilon(u_\varepsilon, w_\varepsilon, \lambda_\varepsilon).
\]
Furthermore,

Assume that $F^*$ is strictly convex and that $\tilde{h}$ is strictly concave in $\mathbb{R}^d/\mathbb{R}$, that is, for all $0 < \mu < 1$ we have

$$\tilde{h}(\mu \Delta_i u + (1 - \mu) \Delta_i v, i) = \mu \tilde{h}(\Delta_i u, i) + (1 - \mu) \tilde{h}(\Delta_i v, i)$$

implies $u = v + k \mathbf{1}$, for some $k \in \mathbb{R}$. Using Jensen’s inequality, we conclude that a solution $(u, w, \lambda) \in \Phi_0$ to

$$\min \left\{ G_0(u, w, \lambda) : (u, w, \lambda) \in \Phi_0 \right\}$$

is such that $(w, u)$ does not depend on time. Thus, in this setting, we establish in addition convergence of solutions of the mean-field game to stationary solutions \textbf{without imposing uniform convexity and uniform monotonicity hypotheses}

\textbf{Thanks for your attention!}