Regularity theory for fully nonlinear elliptic equations

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(based on a joint work with E. Teixeira, UFC and ICMC-USP)

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KAUST, SRI-UQ Seminar Series
March 11th, 2015

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- Final comments and remarks

The Problem and Some Motivation

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where:

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Even worse: given $0 < \tau < 1$ it is possible to build up F_τ, uniformly elliptic such that solutions are not of class $\mathcal{C}^{1,\tau}$.

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Our Result

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Notice that F, F^* and F_μ have the same ellipticity constants.

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Silvestre and Teixeira (14): smoothness of solutions to

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In particular, the authors show that u is of class $\mathcal{C}^{1,Log-Lip}_{loc}(B_1)$.

Result - a priori $W^{2,p}$ estimates

Theorem (P.-Teixeira): Let u be a viscosity solution of $F(D^2u) = f(x)$ with $\|u\|_{L^\infty(B_1)} = 1$. Assume that the recession function F^* associated with F, has $\mathcal{C}^{1,1}$ estimates and that $f \in L^p(B_1)$, for $d < p < \infty$. Then,

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2. $\|u\|_{W^{2,p}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p B_1}\right)$.

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A priori regularity in $W^{2,p}(B_{1/2})$ for solutions.

A few definitions

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$$\underline{G}_M(v,\Omega) \doteq \{x_0 \in \Omega \,|\, \exists P \text{ with } P(x_0) = v(x_0) \text{ and } P \le v\}\,;$$

$$\overline{G}_M(v,\Omega) \doteq \{x_0 \in \Omega \,|\, \exists P \text{ with } P(x_0) = v(x_0) \text{ and } P \ge v\}\,;$$

$$\underline{A}_M(v,\Omega) \doteq \Omega \setminus \underline{G}_M(v,\Omega);$$

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Critical step: to study the decay of $|A_t|$.

A priori regularity in $W^{2,\delta}(B_{1/2})$

Proposition, Lin (86) Let u be a viscosity solution of $F(D^2u) = f(x)$ in B_1 with $\|u\|_{L^\infty(B_1)} = 1$ and $f \in L^d(B_1)$. Then, there exists $\delta > 0$, universal, so that $u \in W^{2,\delta}(B_{1/2})$ and

$$\|u\|_{W^{2,\delta}(B_{1/2})} \leq C \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)} \right).$$

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$$\|u - h\|_{L^\infty(B_{1/2})} \leq \delta.$$

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This inequality allows us to conclude the proof.

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Conjecture: solutions are $p - BMO(B_{1/2})$.

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Thank you Diogo!!