A Stochastic Galerkin Method for Uncertainty Propagation in Conservation Laws

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- Simulation and errors
- Input-data uncertainty

2 Spectral UQ
- Polynomial Chaos expansions
- Application to spectral UQ
- Solution methods

3 Stochastic hyperbolic systems
- Hyperbolic systems
- Galerkin projection
- Approximate Roe Solver

4 Stochastic adaptation
- Tree data structure
- Adaptive scheme
- Burgers equation
- Traffic equation
Simulation framework.

**Basic ingredients**

- Selection of a **mathematical model**:
  retain essential physical processes.

- Selection of a **numerical method**:
  to solve the model equations.

- Define all **input-data** needed:
  select a specific system in the class spanned by the model.

**Simulation errors**

- **Model errors**: physical approximations and simplifications.

- **Numerical errors**: discretization, approximate solvers, finite arithmetics, ...

- **Input-data error**: boundary/initial conditions, model constants and parameters, external forcings, ...
Sources of data uncertainty

- Inherent variability (e.g. industrial processes).
- Epistemic uncertainty (e.g. model constants).
- May not be fully reducible, even theoretically.

Probabilistic framework

- Define an abstract probability space \((\Theta, \mathcal{A}, d\mu)\).
- Consider input-data \(D\) as random quantity: \(D(\theta), \theta \in \Theta\).
- Simulation output \(S\) is random and on \((\Theta, \mathcal{A}, d\mu)\).
- Data \(D\) and simulation output \(S\) are dependent random quantities (through the mathematical model \(\mathcal{M}\)):

\[
\mathcal{M}(S(\theta), D(\theta)) = 0, \quad \forall \theta \in \Theta.
\]
**Propagation of data uncertainty**

Data density

Solution density

\[ \mathcal{M}(S, D) = 0 \]

- **Variability** in model output: numerical error bars.
- Assessment of **predictability**.
- Support **decision making process**.
- **What type of information** (abstract quantities, confidence intervals, density estimations, structure of dependencies, …) one needs?
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Polynomial Chaos expansions

Any well behaved RV $U(\theta)$ (e.g. 2nd-order one) defined on $(\Theta, \mathcal{A}, d\mu)$ has a convergent expansion of the form:

$$U(\theta) = u_0 \Gamma_0 + \sum_{i_1=1}^{\infty} u_{i_1} \Gamma_1(\xi_{i_1}(\theta)) + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} u_{i_1,i_2} \Gamma_2(\xi_{i_1}(\theta), \xi_{i_2}(\theta)) + \cdots$$

- \{\xi_1, \xi_2, \ldots\} : independent normalized Gaussian RVs.
- $\Gamma_p$ polynomials with degree $p$, orthogonal to $\Gamma_q, \forall q < p$.
- Convergence in the mean square sense

[Cameron & Martin, 1947].
Polynomial Chaos expansions

Truncated PC expansion at order $N_0$ and using $N$ RVs:

$$U(\theta) \approx \sum_{k=0}^{P} u_k \psi_k(\xi(\theta)), \quad \xi = \{\xi_1, \ldots, \xi_N\}, \quad P = \frac{(N + N_0)!}{N!N_0!}.$$ 

- $\{u_k\}_{k=0}^{P}$: deterministic expansion coefficients,
- $\{\psi_k\}_{k=0}^{P}$: random polynomials wrt the inner product involving the density of $\xi$:

$$\mathbb{E} \{\psi_k \psi_l\} = \langle \psi_k, \psi_l \rangle \equiv \int_\Theta \psi_k(\xi(\theta))\psi_l(\xi(\theta))d\mu(\theta) = \int \psi_k(\xi)\psi_l(\xi)p(\xi)d\xi = \delta_{kl} \langle \psi_k, \psi_k \rangle.$$ 

- Gaussian RVs: $p(\xi) = \prod_{i=1}^{N} \frac{\exp(-\xi_i^2/2)}{\sqrt{2\pi}} \Rightarrow$ Hermite polynomials (Wiener-Hermite expansions)
- $\{\psi_0, \ldots, \psi_P\}$ is an orthogonal basis of $S^P \subset L^2(\mathbb{R}^N, p(\xi)).$
Polynomial Chaos expansions

Truncated PC expansion:

- Convention $\psi_0 \equiv 1$ : mean mode.
- Expectation of $U$:

$$
\mathbb{E}\{U\} \equiv \int_\Theta U(\theta) d\mu(\theta) \approx \sum_{k=0}^P u_k \int_\Xi \psi_k(\xi) p(\xi) d\xi = u_0.
$$

- Variance of $U$:

$$
V[U] = \mathbb{E}\{U^2\} - \mathbb{E}\{U\}^2 \approx \sum_{k=1}^P u_k^2 \langle \psi_k, \psi_k \rangle.
$$

- Extension to random vectors & stochastic processes:

$$
\begin{pmatrix}
U_1 \\
\vdots \\
U_m
\end{pmatrix}
(\theta, x, t) \approx \sum_{k=0}^P
\begin{pmatrix}
u_1 \\
\vdots \\
u_m
\end{pmatrix}_k (x, t) \psi_k(\xi(\theta)).
$$
Generalized PC expansion

Askey scheme

<table>
<thead>
<tr>
<th>Distribution of $\xi_i$</th>
<th>Polynomial family</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>Hermite</td>
</tr>
<tr>
<td>Uniform</td>
<td>Legendre</td>
</tr>
<tr>
<td>Exponential</td>
<td>Laguerre</td>
</tr>
<tr>
<td>$\beta$-distribution</td>
<td>Jacobi</td>
</tr>
</tbody>
</table>

Also: discrete RVs (Poisson process).

$$U(\theta) \approx \sum_{k=0}^{P} u_k \psi_k(\xi(\theta))$$

where $\psi_k$: classical (or product of) polynomials:

spectral expansions
Instead of a spectral expansion over $\Xi$ one can use **Piecewise polynomial expansion** on a mesh $\Sigma$ of $\Xi$

- $\Xi = \bigcup_{SE \in \Sigma} \Xi_{SE}$, $\Xi_{SE} \cap \Xi_{SE'} = \emptyset$ for $SE \neq SE'$
- $S = \left\{ U \in L^2(\Xi, p_\xi), U(\xi \in \Xi_{SE}) \in P^N_{\text{No}}(\Xi_{SE}) \right\}$

$$U(\theta) \approx \sum_{k=0}^{P} u_k \psi_k(\xi(\theta))$$

- $\psi_k$ are orthogonal with:
  1. Support of $\psi_k$ limited to an element: **Stochastic multi-element method** [Deb et al., 2001], [Wang and Karniadakis, 2005]
    
    *Fully decouple the approximation problems over different elements*
  2. Hierarchical orthogonal functions $\psi_k$: **Stochastic Multiwavelet method** [olm et al., 2004, 2006, 2009]
    
    *Coupled problems, well suited for adaptive strategy (MRA)*
Input-data parametrization

Parametrization of $D$ using $N < \infty$ independent RVs with prescribed distribution $p(\xi)$:

$$D(\theta) \approx D(\xi(\theta)), \quad \xi = (\xi_1, \ldots, \xi_N) \in \Xi.$$ 

- Iso-probabilistic transformations of RVs,
- Karhunen-Loève expansion: $D(x, \theta)$ stochastic field/process,
- Identification (e.g. Bayesian).

Model

Solution expansion
Input-data parametrization

Model

We assume that for a.e. $\xi \in \Xi$, the problem $\mathcal{M}(S, D(\xi)) = 0$

1. is well-posed,
2. has a unique solution

and that

the random solution $S(\xi) \in L^2(\Xi, p_\xi)$:

$$\mathbb{E} \left\{ S^2 \right\} = \int_\Theta S^2(\xi(\theta)) d\mu(\theta) = \int_{\Xi} S^2(\xi) p(\xi) d\xi < +\infty.$$
Input-data parametrization

Model

Solution expansion

Let \( \{\psi_0, \psi_1, \ldots\} \) be a basis of \( L^2(\Xi, p_\xi) \) then

\[
S(\xi) = \sum_k s_k \psi_k(\xi).
\]

- Knowledge of the spectral coefficients \( s_k \) fully determine the random solution.
- Makes explicit the dependence between \( D(\xi) \) and \( S(\xi) \).
Input-data parametrization

Model

Solution expansion

Let \( \{ \psi_0, \psi_1, \ldots \} \) be a basis of \( L^2(\Xi, p_\xi) \) then

\[
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\]

- Knowledge of the spectral coefficients \( s_k \) fully determine the random solution.
- Makes explicit the dependence between \( D(\xi) \) and \( S(\xi) \).
- Need efficient procedure(s) to compute the \( s_k \).
Galerkin projection

1. Introduce truncated expansions in model equations
2. Require residual to be $\perp$ to the stochastic subspace $S^p$

\[
\left\langle M\left(\sum_{k=0}^{P} s_k \psi_k(\xi), D(\xi)\right), \psi_m(\xi) \right\rangle = 0 \quad \text{for } m = 0, \ldots, P.
\]

Set of $P + 1$ coupled problems.

**Plus**
- Implicitly account for modes’ coupling
- Often inherit properties of the deterministic model

**Minus**
- Requires adaptation of deterministic solvers
- Treatment of non-linearities.
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PhD work of Julie Tryoen

with Alexandre Ern (Cermics, Univ. Paris-Est) and Michael Ndjinga (CEA, Saclay)
Hyperbolic systems: deterministic case

\[ \frac{\partial u}{\partial t} + \nabla \cdot f(u) = 0, \quad u(x, t = 0) = u^0(x), \quad BCs \]

\( u \in A_u \subset \mathbb{R}^m \) (conservative variables)

\( f : A_u \mapsto \mathbb{R}^m \) (flux function)

\( \text{if } \nabla uf \in \mathbb{R}^{m \times m} \text{ is } \mathbb{R}\text{-diagonalizable on } A_u \Longrightarrow \text{hyperbolic} \)

\( u \) can develop shocks / discontinuities in finite time

Classical discretization (Finite Volume in 1-space dimension)

\[ \frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{\tilde{f}(u_i^n, u_{i+1}^n) - \tilde{f}(u_{i-1}^n, u_i^n)}{\Delta x} = 0 \]

where \( u_i^n = \int_{\Delta x} u(x, t_n)dx \) and \( \tilde{f}(, ) \) is the numerical flux function (having had-hoc properties).
Uncertain hyperbolic problems:

- Uncertain initial & boundary conditions and parameters in $f$
- Parametrization with $\xi(\theta) = \{\xi_1(\theta), \ldots; \xi_N(\theta)\}$ a set of $N$ iid random variables with uniform distribution on $\Xi = [0, 1]^N$
- Stochastic Hyperbolic Problem

$$\frac{\partial U(x, t, \xi)}{\partial t} + \nabla \cdot F(U; \xi) = 0, \quad U(x, t = 0, \xi) = U^0(x, \xi) \quad (a.s.)$$

Hypotheses

1. $U(x, t, \xi) \in \mathcal{A}_U$ and $\nabla_u F(U; \xi)$ is $\mathbb{R}$-diagonalizable a.s.
2. All random quantities have finite variance.

The solution is sought in $S^p := \text{span} \{\psi_0, \ldots, \psi_P\} \subset L_2(\Xi)$, where $\psi_\alpha$ are orthonormal polynomials in $\xi$ with degree $\leq N_0 : \langle \psi_\alpha, \psi_\beta \rangle = \delta_{\alpha, \beta}$. 
Galerkin problem:

- Since $U \in L^2(\Omega)$ it has a convergent expansion:
  \[ U(x, t, \xi) = \sum_\alpha u_\alpha(x, t)\psi_\alpha(\xi) \]

- We denote $U^P$ the approximation of $U$ in $S^P$

- Stochastic Galerkin projection of the hyperbolic problem: for $\alpha = 0, \ldots, P$
  \[
  \frac{\partial u_\alpha(x, t)}{\partial t} + \nabla \cdot f_\alpha(u_0, \ldots, u_P) = 0
  
  f_\alpha(u_0, \ldots, u_P) \equiv \langle F(U^P; \xi), \psi_\alpha \rangle
  
  u_\alpha(x, t = 0) = \langle U^0(x), \psi_\alpha \rangle
  
  (P + 1)$-coupled problems for the solution modes
Galerkin problem : (system form)

\[
\frac{\partial}{\partial t} \begin{pmatrix} u_0 \\ \vdots \\ u_P \end{pmatrix} + \nabla \cdot \begin{pmatrix} f_0(u_0, \ldots, u_P) \\ \vdots \\ f_P(u_0, \ldots, u_P) \end{pmatrix} = 0
\]

\[
\frac{\partial U}{\partial t} + \nabla \cdot \mathcal{F}(U) = 0
\]

- \( U \in \mathbb{R}^{m \times (P+1)} \)
- \( \mathcal{F} : \mathbb{R}^{m \times (P+1)} \rightarrow \mathbb{R}^{m \times (P+1)} \)
- Is the Galerkin problem hyperbolic?
- \( (\nabla_U \mathcal{F} \ \mathbb{R}\text{-diagonalizable?}) \)
- What is the admissible domain \( \mathcal{A}_U \)?
Jacobian of the Galerkin problem

\[
\nabla_u F = \begin{pmatrix}
F_{0,0}' & \ldots & F_{0,P}' \\
\vdots & \ddots & \vdots \\
F_{P,0}' & \ldots & F_{P,P}'
\end{pmatrix}, \quad F_{\alpha,\beta}' = \left\langle \nabla u F(U^P; \xi), \psi_\alpha \psi_\beta \right\rangle \in \mathbb{R}^{m,m}
\]

- If \( \nabla_u F \) is symmetric (a.s.), \( \nabla_u F \) is \( \mathbb{R} \)-diagonalizable
- In particular, scalar problems (\( m = 1 \)) yield hyperbolicity
- If \( \nabla_u F = LD(\xi)R \), where \( L \) and \( R \) are deterministic, the Galerkin problem is hyperbolic
- Properties extend to \( \neq \) truncature rules
- Note that strict hyperbolicity is \textbf{not} to be expected even when \( \nabla_u F \) has (a.s.) distinct eigenvalues.

\[\text{[J. Tryoen et al, JCP 2010, JCAM 2010]}\]
General case
Let \( \{\xi^{(i)}\} \) and \( \{w^{(i)}\} \), \( i = 0, \ldots, P \) the points and weights of the Gauss’ quadrature on \( \Xi \).
Define
\[
(\nabla u \mathcal{F})_{\alpha,\beta} = \sum_{i=0}^{P} \nabla u \mathcal{F} \left( U^P(\xi^{(i)}); \xi^{(i)} \right) \psi_{\alpha}(\xi^{(i)}) \psi_{\beta}(\xi^{(i)}) w^{(i)} \approx \mathcal{F}'_{\alpha,\beta}
\]
\( \Rightarrow \) \( \nabla u \mathcal{F} \) is \( \mathbb{R} \)-diagonalizable
\( \Rightarrow \) Let \( \{\Lambda_l(\xi)\}_{l=1}^{m} \), the stochastic Eigenvalues of \( \nabla \mathcal{F} \)
\( \{\Lambda_l \equiv \Lambda_l(\xi^{(i)})\} \) are the eigenvalues of \( \nabla u \mathcal{F} \)
\( \Rightarrow \) For sufficient smoothness, \( \lim_{N_0 \to \infty} \nabla u \mathcal{F} = \nabla u \mathcal{F} \)
\( \Rightarrow \) Use \( \{\Lambda_l(\xi^{(i)})\} \) as approximate spectrum of \( \nabla u \mathcal{F} \)

[J. Tryoen et al, JCP 2010, JCAM 2010]
Approximate Roe solver

\[
U_{i}^{n+1} = U_{i}^{n} - \frac{\Delta t}{\Delta x} \left[ \phi(U_{i}^{n}, U_{i+1}^{n}) - \phi(U_{i-1}^{n}, U_{i}^{n}) \right]
\]

where the numerical flux \( \Phi \) is chosen as

\[
\phi(U_{L}, U_{R}) = \frac{1}{2} \left[ \mathcal{F}(U_{L}) + \mathcal{F}(U_{R}) \right] - a \frac{U_{R} - U_{L}}{2}
\]

where \( a \in \mathbb{R}^{m(P+1) \times m(P+1)} \) is a non-negative upwind matrix

**Theorem**: It exists a Galerkin Roe state \( U_{RL}^{Roe} \) such that \( \nabla \mathcal{F}(U_{RL}^{Roe}) \) is a Roe matrix for the Galerkin problem i.e. has properties of consistency and conservativity through shocks.

We will take

\[
\phi(U_{L}, U_{R}) = \frac{1}{2} \left[ \mathcal{F}(U_{L}) + \mathcal{F}(U_{R}) \right] - \left| \nabla_{u} \mathcal{F}(U_{RL}^{Roe}) \right| \frac{U_{R} - U_{L}}{2}
\]

where \( |A| = |LDR| = L |D| R \) for a \( \mathbb{R} \)-diagonalizable matrix

\[\text{[J. Tryoen et al, JCP 2010]}\]
Fast approximation of the upwind matrix

To avoid the costly decomposition of the Roe matrix, we rely on a polynomial transform $q_d$:

- recall $q(LDR) = Lq(D)R$
- $|\mathcal{N}_U F| \approx q_d (\mathcal{N}_U F)$, where $q_d \in \mathbb{P}_d$ minimizes
  \[
  J = \sum_{i,l} \left[ q_d \left( \Lambda^l_i \right) - \Lambda^l_i \right]^2, \quad \Lambda^l_i \approx \Lambda^l \left( U_{LR}^\text{Roe} (\xi^{(i)}) \right)
  \]

- In practice $d \leq 6$ is sufficient

Approximation polynomial $q_d$ for $d = 2$ (left) and $d = 6$ (right).
Summary:

\[
U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} \left[ \phi(U_i^n, U_{i+1}^n) - \phi(U_{i-1}^n, U_i^n) \right]
\]

where

\[
\phi(U_L, U_R) = \frac{1}{2} \left[ \mathcal{F}(U_L) + \mathcal{F}(U_R) \right] - q_d \left( \nabla U \mathcal{F}(U_{\text{Roe}}) \right) \frac{U_R - U_L}{2}
\]

- Upwinding w.r.t. the actual waves in the Galerkin solution
- Applies conditionally to partially tensored stochastic basis
- May need Entropy corrector
- Assume \( U(\xi) \) smooth and sufficient stochastic discretization
- But solutions are not smooth in general!

Call for piecewise polynomial approximations to allow for discontinuities at the stochastic level
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Binary trees for piecewise polynomial space

- Dyadic partitions of a node along a prescribe direction $d : p \rightarrow (c^-, c^+)$
- Piecewise-polynomial with fixed order $N_0$ on each leaf of $T$.

1. Union of local modal basis : SE-basis
   [Deb et al, 2001], [Karniadakis et al]
   Uncoupled application of the Roe scheme over different leafs

2. Hierarchical global basis over $\Xi$ : MW-Basis
   [OLM et al, 2004]
   Hierarchical sequence of details, suited for adaptive scheme
Adaptivity
Singularity curves are localized in $\Xi$: stochastic adaptivity

- Incomplete and anisotropic binary trees

Operators for multi-resolution analysis:

- **Prediction operator**: define the solution in a stochastic space larger than the current one (add new leafs and $L^2$-injection).

- **Restriction operator**: define the solution in a stochastic space smaller one the current one (remove leafs and $L^2$-projection).

- Rely on recursive application of **elementary (directional) operators**, full exploitation of the tree structure.
Adaptivity:

Singularity curves are localized in $x$ and $t$

- Each spatial cell carries its own adapted stochastic discretization
- Flux computation,
  \[
  \phi(U_L, U_R) = \frac{F(U_L) + F(U_R)}{2} - |a^{\text{Roe}}(U_L, U_R)| \frac{U_R - U_L}{2},
  \]
  with $U_R$ and $U_L$ known on different stochastic spaces
- Union operator: given two stochastic spaces, construct the minimal stochastic space containing the two:
Adaptive Algorithm :

1. Loop over all interfaces of the spatial mesh :
   - Construct the union space of the left and right cells
   - **Enrich** this space
   - **Predict** left and right states of the interface
   - Evaluate the numerical flux (App. Roe scheme)

2. Loop over all cells of the spatial mesh :
   - Construct the union space of the cell’s interfaces
   - **Predict** cell’s fluxes on the union space
   - Compute fluxes difference and update cell’s solution
   - **Restrict** cell’s solution by **thresholding**

3. Repeat for the next time step

**Two indicators needed** : based on multiwavelet details of nodes.

- for **Enrichment** : anticipate emergence of new stochastic details,
- for **Thresholding** : remove unnecessary/negligible details.
Thresholding criterion:
Let us denote
- $T$ a binary tree and $S(T)$ the corresponding stochastic approximation space
- $n \in \mathcal{N}(T)$ a node of the tree, and $\mathcal{N}(T)$ set set of nodes having children
- $N_r$ the maximal depth allowed in a direction
- $T_{[NN_r]}$ the maximal tree given $N_r$

We define for $U \in S(T_{[NN_r]})$ and $\eta > 0$ the subset of $\mathcal{N}(T_{[NN_r]})$

$$D(\eta) := \left\{ n \in \mathcal{N}(T_{[NN_r]}); \| \tilde{u}^n \|_{\ell^2} \leq 2^{-|n|/2} \frac{\eta}{\sqrt{NN_r}} \right\},$$

where $\tilde{u}^n := (\tilde{u}_\alpha^n)_{1 \leq \alpha \leq p}$ are the MW coefficients of $n$. 
Coarsening strategy:
Two sisters \((c^-, c^+)\) of a parent \(p(c^-)\) are removed from the discretization if

\[
\| \tilde{u}^{p(c^-)} \|_{L^2} \leq 2^{-|n|/2} \frac{\eta}{\sqrt{NNr}}
\]

The criterion ensures that

\[
\| U^{T[NNr]} - U^{T[NNr]}_D \| \leq \eta.
\]

<table>
<thead>
<tr>
<th>(\alpha = 1)</th>
<th>(\alpha = 2)</th>
<th>(\alpha = 3)</th>
<th>(\alpha = 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Mother wavelets for N = 2, No = 1 in direction d = 1." /></td>
<td><img src="image2" alt="Mother wavelets for N = 2, No = 1 in direction d = 1." /></td>
<td><img src="image3" alt="Mother wavelets for N = 2, No = 1 in direction d = 1." /></td>
<td><img src="image4" alt="Mother wavelets for N = 2, No = 1 in direction d = 1." /></td>
</tr>
</tbody>
</table>

Mother wavelets \(\tilde{\Psi}^d_\alpha\) for \(N = 2, No = 1\) in direction \(d = 1\).

Note: the coarsening is applied to the class of equivalent trees.
Enrichment strategy:
Enrichment is necessary to anticipate emergence of new-stochastic details.

- Isotropic enrichment is not an option for $N > 2, 3$
- 1-D enrichment criterion: if $U$ is (locally) smooth enough $\tilde{u}_\alpha^n$ of a generic node $n$ can be bounded as

$$|\tilde{u}_\alpha^n| = \inf_{P \in \mathbb{P}_N[\xi]} |\langle (U - P), \psi^n_\alpha \rangle| \leq C |S(n)|^{N+1} \|U\|_{H^{N+1}(S(n))},$$

where $|S(n)| = 2^{-|n|}$ is the volume of the node. Therefore

$$\|\tilde{u}^n\|_{\ell^2} \sim 2^{-(N+1)} \|\tilde{u}^{p(n)}\|_{\ell^2}$$

and a leaf $l$ is refined if

$$\|\tilde{u}^{p(l)}\|_{\ell^2} \geq 2^{N+1} 2^{-|l|/2} \eta/\sqrt{N_r} \quad \text{and} \quad |l| < N_r.$$
Enrichment strategy:
Extension of to the $N$-dimensional case:

- Using the decay estimation

\[ |\tilde{u}_\alpha^n| = \inf_{P \in \mathbb{P}_{No}^N} \left| \langle (U - P), \psi_{\alpha}^{n,d} \rangle \right| \leq C \text{diam}(S(n))^{No+1} \|U\|_{H^{No+1}(S(n))}, \]

- a leaf $l$ is partitioned in direction $d$ if

\[ \|\tilde{u}_{P_{d}(l)}\|_{\ell^2} \geq \frac{\text{diam}(S(P_{d}(l)))^{No+1}}{\text{diam}(S(l))} \leq 2^{-|l|/2} \eta/\sqrt{NNr} \text{ and } |S(l)|_d > 2^{-Nr}. \]

- the construction of the virtual sister and parent of $l$ in arbitrary direction $d$

A sharper anisotropic criterion has been proposed using $1 - D$ analysis functions in direction $d$ [J.Tryoen, preprint 2012].
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Burgers equation

\[
\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2}
\]

Uncertain initial condition \( U^0(x, \xi) : \)

\[
X_{1,2} = 0.1 + 0.1\xi_1, \quad X_{2,3} = 0.3 + 0.1\xi_2, \quad \xi_1, \xi_2 \sim \mathcal{U}[0, 1]
\]

2 stochastic dimensions.
**Burgers equation**

\[
\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2}
\]
**Burgers equation**

\[
\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2}
\]
Trafic equation in periodic $[0, 1]$-domain

$$F(U(\xi); \xi) = A(\xi)U(\xi)(1 - U(\xi))$$
1-Periodic BC.

- uncertain initial density of vehicles

$$U^0(x, \xi) = 0.25 + 0.01\xi_1 - I_{[0.1,0.3]}(x)(0.2 + 0.015\xi_2) + I_{[0.3,0.5]}(x)(0.1 + 0.015\xi_3) - I_{[0.5,0.7]}(x)(0.2 + 0.015\xi_4)$$

- uncertain characteristic velocity $A(\xi) = 1 + 0.1\xi_5$

- 5-dimensional problem $(\xi_1, \ldots, \xi_5) \sim U[0, 1]^5$.

20 realizations of the initial condition (left) and solution at $t = 0.4$ (middle) and $t = 0.9$ (right): 2 shocks and 2 rarefaction waves.
Space-time diagrams of the solution mean (left), standard deviation (center) and average depth of the leaves (right):

Averaged number of partitions in each direction $D_i$ and anisotropy factor $\rho$:

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<tr>
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<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>$D_5$</th>
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<td><img src="image1" alt="Diagram" /></td>
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</table>
Hoeffding decomposition.

$$U(\xi_1, \ldots, \xi_N) = U_0 + \sum_{i_1=1}^{N} U_{i_1}(\xi_{i_1}) + \sum_{i_1=1}^{N} \sum_{i_2=i_1+1}^{N} U_{i_1,i_2}(\xi_{i_1}, \xi_{i_2}) + \ldots + U_{1,\ldots,N}(\xi_{i_1}, \ldots, \xi_{i_N})$$

Orthogonal hierarchical decomposition

Sobol ANOVA (analysis of the variance)

$$V(U) = \sum_{i_1=1}^{N} V_{i_1} + \sum_{i_1=1}^{N} \sum_{i_2=i_1+1}^{N} V_{i_1,i_2} + \cdots + V_{1,\ldots,N},$$

- First order sensitivity indexes: $$S_i = \frac{V_i}{V}$$
- Total sensitivity indexes: $$T_i = \sum_{u \supseteq \{i\}}^{u \ni \{1,\ldots,N\}} \frac{V_u}{V}$$
### Space-time diagrams of the 1-st order sensitivity indexes $S_i$ and contribution of higher order indexes.
Total sensitivity indices as a function of $x \in [0, 1]$ at $t = 0.4$ (left) and $t = 0.9$ (right).
$L^2$-norm of stochastic error for different values of $\eta \in [10^{-2}, 10^{-5}]$ and polynomial degrees $N_0$

Left: error as a function of the total number of leafs in the final discretization ($t^n = 0.5$). Right: error as a function of the total number of degrees of freedom (number of leafs times the dimension of the local polynomial basis).
Computational time (per time-iteration) as a function of the stochastic discretization (total number of leafs); left: $N_0 = 2$ and $\eta = 10^{-3}$; right: $N_0 = 3$ and $\eta = 10^{-4}$.
-Thank you for your attention-

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2nd test case
Continuous initial conditions : two constants stochastic states

\[ U = U^+ = 1 \pm 0.05 \quad x < 1/3, \]
\[ U = U^- = -1 \pm 0.1 \quad x > 2/3, \]

and affine variation in between. \( U^+ > U^- \) a.s. and \( U^+ \) and \( U^- \) independent with uniform distribution : \( U^+(\xi_1), \ U^-(\xi_2) \).

\[ U^+(\xi_1) + U^-(\xi_2) \neq 0 \text{ a.s.} \]
2nd test case

Solution with $x$ at different times.
Convergence with max resolution level (8 to 14)