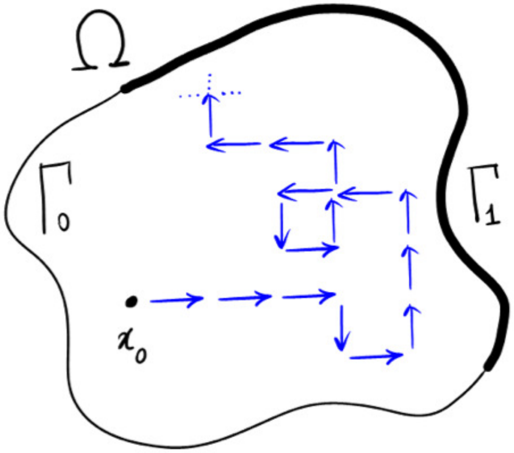
Game Theoretical Methods in PDEs

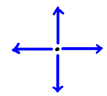
Tutorial

Marta Lewicka

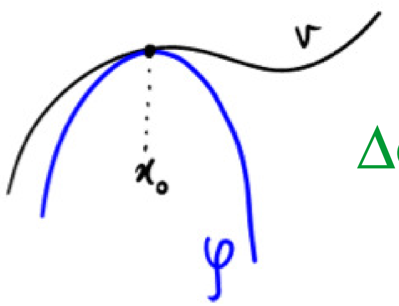
University of Pittsburgh

Linear PDEs (Δ) and probability

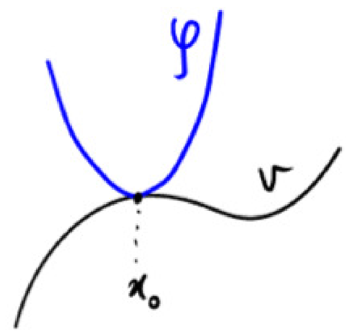


- Initial position of token: $x_0 \in \Omega \subset \mathbb{R}^2$.
- Moves:  random increments of length ε.
- $u_\varepsilon(x_0)$: probability of hitting $O_\varepsilon(\Gamma_1)$,
  the first time $O_\varepsilon(\partial\Omega)$ is hit.

<u>THM</u> $u_\varepsilon \rightrightarrows v \in C(\Omega)$ as $\varepsilon \to 0$, and v is a viscosity solution to $\Delta v = 0$:



$\Delta\phi(x_0) \leq 0$



$\Delta\phi(x_0) \geq 0$

In fact, v is the classical solution to:

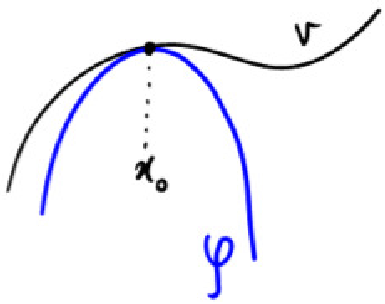
$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v = 1 & \text{on } \Gamma_1 \quad \text{and} \quad v = 0 \text{ on } \partial\Omega \setminus \Gamma_1 \end{cases}$$

Proof that v is a viscosity solution

$$u_\varepsilon(x) = \tfrac{1}{4}\big(u_\varepsilon(x+\varepsilon e_1) + u_\varepsilon(x-\varepsilon e_1) + u_\varepsilon(x+\varepsilon e_2) + u_\varepsilon(x-\varepsilon e_2)\big)$$

$$\Leftrightarrow \quad 0 = \big(u_\varepsilon(x+\varepsilon e_1) - u_\varepsilon(x)\big) + \big(u_\varepsilon(x-\varepsilon e_1) - u_\varepsilon(x)\big)$$
$$+ \big(u_\varepsilon(x+\varepsilon e_2) - u_\varepsilon(x)\big) + \big(u_\varepsilon(x-\varepsilon e_2) - u_\varepsilon(x)\big)$$

- $u_\varepsilon \rightrightarrows v$ in $\bar{\Omega}$ as $\varepsilon \to 0$ (can be proven rigorously).



Take ϕ smooth, $(v-\phi)(x_0) = 0 \le (v-\phi)(x)\ \ \forall x$.

Find $x_\varepsilon \to x_0$: $(u_\varepsilon - \phi)(x_\varepsilon) \le (u_\varepsilon - \phi)(x) + \varepsilon^3 \quad \forall x$

$$\Leftrightarrow \quad u_\varepsilon(x) - u_\varepsilon(x_\varepsilon) \ge \phi(x) - \phi(x_\varepsilon) - \varepsilon^3$$

- $0 \ge \big(\phi(x_\varepsilon+\varepsilon e_1) - \phi(x_\varepsilon)\big) + \big(\phi(x_\varepsilon-\varepsilon e_1) - \phi(x_\varepsilon)\big)$
$$+ \big(\phi(x_\varepsilon+\varepsilon e_2) - \phi(x_\varepsilon)\big) + \big(\phi(x_\varepsilon-\varepsilon e_2) - \phi(x_\varepsilon)\big) - 4\varepsilon^3$$

(Taylor) $= \varepsilon^2 \partial_{11}\phi(x_\varepsilon) + \varepsilon^2 \partial_{22}\phi(x_\varepsilon) + o(\varepsilon^2) = \varepsilon^2 \Delta\phi(x_\varepsilon) + o(\varepsilon^2)$

- Conclude: $0 \ge \Delta\phi(x_\varepsilon) + o(1)$. Hence: $\Delta\phi(x_0) \le 0$.

Non-linear PDEs (Δ_p)

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2}\Delta u + \left\langle \nabla(|\nabla u|^{p-2}) : \nabla u \right\rangle$$
$$= |\nabla u|^{p-2}\left(\Delta u + (p-2)\langle \nabla^2 u : \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|}\rangle\right)$$
$$= |\nabla u|^{p-2}(\Delta u + (p-2)\ \Delta_\infty u)$$

- Taylor expand: $u(x) = u(x_0) + \langle \nabla u(x_0), x - x_0\rangle$
$$+ \tfrac{1}{2}\langle \nabla^2 u(x_0) : (x-x_0)\otimes(x-x_0)\rangle + o(\varepsilon^2)$$

Average: $\fint_{B_\varepsilon(x_0)} u = u(x_0) + \frac{1}{2}\langle \nabla^2 u(x_0) : \fint (x-x_0)\otimes(x-x_0)\rangle$
$$= u(x_0) + \tfrac{1}{2}\langle \nabla^2 u(x_0) : \tfrac{\varepsilon^2}{4}\,\mathrm{Id}\rangle = u(x_0) + \tfrac{\varepsilon^2}{8}\,\Delta u + o(\varepsilon^2)$$

- **Thm** (Manfredi-Parviainen-Rossi 2012)

  Let $u \in \mathcal{C}(\Omega)$ satisfy: $\forall x_0 \in \Omega \quad u(x_0) = \fint_{B_\varepsilon(x_0)} u + o(\varepsilon^2)$. Then $\Delta u = 0$ in Ω.

- $\frac{1}{2}(\sup_{B_\varepsilon(x_0)} u + \inf_{B_\varepsilon(x_0)} u) \approx \frac{1}{2}\left(u(x_0 + \varepsilon\frac{\nabla u(x_0)}{|\nabla u(x_0)|}) + u(x_0 - \varepsilon\frac{\nabla u(x_0)}{|\nabla u(x_0)|})\right)$
$$= u(x_0) + \frac{\varepsilon^2}{2}\left\langle \nabla^2 u(x_0) : \frac{\nabla u(x_0)}{|\nabla u(x_0)|} \otimes \frac{\nabla u(x_0)}{|\nabla u(x_0)|}\right\rangle + o(\varepsilon^2)$$
$$= u(x_0) + \frac{\varepsilon^2}{2}\,\Delta_\infty u(x_0) + o(\varepsilon^2)$$

Non-linear PDEs (Δ_p)

$$\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2}\Delta u + \left\langle \nabla(|\nabla u|^{p-2}) : \nabla u \right\rangle$$
$$= |\nabla u|^{p-2}\left(\Delta u + (p-2)\langle \nabla^2 u : \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|}\rangle\right)$$
$$= |\nabla u|^{p-2}(\Delta u + (p-2)\ \Delta_\infty u)$$

- We have: $\fint_{B_\varepsilon(x_0)} u = u(x_0) + \frac{\varepsilon^2}{8}\,\Delta u + o(\varepsilon^2)$
- $\frac{1}{2}(\,\sup_{B_\varepsilon(x_0)} u + \inf_{B_\varepsilon(x_0)} u) = u(x_0) + \frac{\varepsilon^2}{2}\,\Delta_\infty u(x_0) + o(\varepsilon^2)$

Hence: $\frac{4}{p+2} \fint_{B_\varepsilon(x_0)} u + \frac{p-2}{p+2}\,\frac{1}{2}(\,\sup_{B_\varepsilon(x_0)} u + \inf_{B_\varepsilon(x_0)} u)$
$$= u(x_0) + \frac{\varepsilon^2}{2(p+2)}(\Delta u + (p-2)\Delta_\infty u) + o(\varepsilon^2)$$

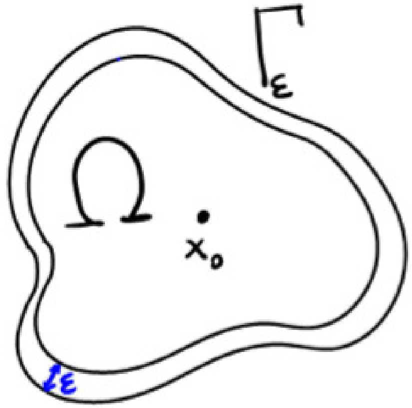
Concluding: If $\Delta_p u = 0$ then:

$$\boxed{u(x_0) = \frac{\alpha}{2}\sup_{B_\varepsilon(x_0)} u + \frac{\alpha}{2}\inf_{B_\varepsilon(x_0)} u + \beta \fint_{B_\varepsilon(x_0)} u + o(\varepsilon^2)}$$

$(\Omega \subset \mathbb{R}^2,\ \ 2 \le p < \infty)$ $\qquad$ $(\alpha = \frac{p-2}{p+2},\ \ \alpha + \beta = 1)$

Tug-of-War game for Δ_p

$X = \Omega \cup \Gamma_\varepsilon.$



Rules of the game: • Token initially at $x_0 \in \Omega$.

- probability $\frac{\alpha}{2} \to$ player I moves by at most ε
- probability $\frac{\alpha}{2} \to$ player II moves by at most ε
- probability $\beta \to$ computer

moves randomly by at most ε

Game stops, when token position $x_n \in \Gamma_\varepsilon$

Then player II pays $F(x_n)$ to player I. Boundary data: $F : \Gamma_\varepsilon \to \mathbb{R}$.

Strategies: $\sigma_I = \{\sigma_I^k : X^{k+1} \to X\}_{k=1}^{\infty}$, $\sigma_{II} = \{\sigma_{II}^k : X^{k+1} \to X\}_{k=1}^{\infty}$

Borel functions, $\sigma_{I,II}^k(x_0, \ldots x_k) \in B_\varepsilon(x_k)$,

and $\sigma_{I,II}^k(x_0, \ldots x_k) = x_k$ if $x_k \in \Gamma_\varepsilon$.

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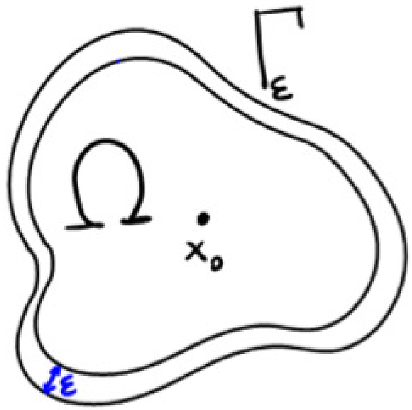
Tug-of-War game for Δ_p

$X = \Omega \cup \Gamma_\varepsilon$.



Strategies: $\sigma_{I,II} = \{\sigma^k_{I,II} : X^{k+1} \to X\}_{k=1}^{\infty}$,

Probability measures:

$$\gamma_k[x_0,\dots x_k] = \frac{\alpha}{2}\delta_{\sigma^k_I(x_0,\dots x_k)} + \frac{\alpha}{2}\delta_{\sigma^k_{II}(x_0,\dots x_k)} + \beta \begin{cases} \frac{\mathcal{L}\lfloor B_\varepsilon(x_k)}{|B_\varepsilon|} & \text{if } x_k \in \Omega \\ \delta_{x_k} & \text{if } x_k \in \Gamma_\varepsilon \end{cases}$$

- Probabilities $\mathbb{P}^{x_0,k}_{\sigma_I,\sigma_{II}}$:

$$\mathbb{P}^{x_0,k}_{\sigma_I,\sigma_{II}}(A_1 \times \dots A_k) = \int_{A_1} \dots \int_{A_k} 1 \, \mathrm{d}\gamma_{k-1}[x_0,\dots x_{k-1}] \, \dots \, \mathrm{d}\gamma_0[x_0].$$

- They generate (Kolmogoroff extension) probability $\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II}}$

  on sets of histories $(x_0, x_1, \dots) \in X^\infty$.

- Define the expectation: $\boxed{\mathbb{E}^{x_0}_{\sigma_I,\sigma_{II}}[F_\tau] = \int_{X^\infty} F(x_\tau) \, \mathrm{d}\mathbb{P}^{x_0}_{\sigma_I,\sigma_{II}}}$

where $\tau = \min\{k;\ x_k \in \Gamma_\varepsilon\}$ and $F(x_\tau) = \begin{cases} F(x_k) & \text{if } \tau = k \\ +\infty & \text{if } \tau = +\infty \end{cases}$

Tug-of-War game for Δ_p

minimum gain of player I: $u_I(x_0) = \sup_{\sigma_I} \inf_{\sigma_{II}} \mathbb{E}^{x_0}_{\sigma_I,\sigma_{II}}[F_\tau]$

maximum loss of player II: $u_{II}(x_0) = \inf_{\sigma_{II}} \sup_{\sigma_I} \mathbb{E}^{x_0}_{\sigma_I,\sigma_{II}}[F_\tau]$

THM 1 (Peres-Schramm-Sheffield-Wilson 2008 ($p = \infty$))
(Peres-Sheffield 2008 ($1 < p < \infty$))
(Manfredi-Parviainen-Rossi 2012 ($2 \leq p < \infty$))

$u_I = u_{II} = u_\varepsilon :=$ the unique p-harmonious function:

$$\begin{cases} u(x) = \dfrac{\alpha}{2} \sup\limits_{B_\varepsilon(x)} u + \dfrac{\alpha}{2} \inf\limits_{B_\varepsilon(x)} u + \beta \fint_{B_\varepsilon(x)} u & \text{in } \Omega \\ u = F & \text{in } \Gamma_\varepsilon \end{cases}$$

THM 2 When $\varepsilon \to 0$ then $u_\varepsilon \rightrightarrows v$ the unique viscosity solution to:

$$\begin{cases} \Delta_p v = 0 & \text{in } \Omega \\ u = F & \text{on } \partial\Omega \end{cases}$$

Plan for the lecture

- complete proofs of Theorem 1 and Theorem 2
- proof of the Harnack inequality for $\Delta_p v = 0$,

  using tug-of-war game interpretation
- limiting cases $p \to \infty$ and $p \to 1$
  - $p = \infty$: infinity Laplacian $\Delta_\infty v = 0$

    and absolutely minimizing Lipschitz extensions
  - $p = 1$: motion by mean curvature $v_t - |\nabla v| \text{div}(\frac{\nabla v}{|\nabla v|}) = 0$

    and "geometric" equations, Kohn-Serfaty game

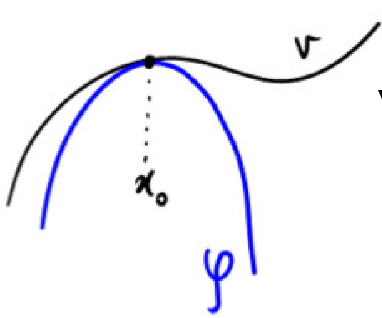
Basic facts on $\Delta_p v = \textbf{div}(|\nabla v|^{p-2}\nabla v)$, $1 < p < \infty$

- $p = 2$: Laplacian, $p > 2$: degenerate elliptic, $1 < p < 2$: singular
- Euler-Lagrange eqn for: $I_p(v) = \frac{1}{p}\int_\Omega |\nabla v|^p$ on $v \in W^{1,p}(\Omega)$

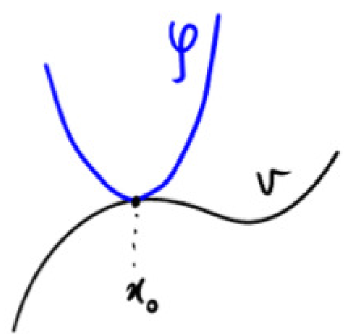
  $\Rightarrow$ existence and uniqueness of a weak solution to: $\begin{cases} \Delta_p v = 0 \\ v_{|\partial\Omega} = g \end{cases}$
- weak solutions are $C^{1,\alpha}_{loc}$, in general not C^2 when $\nabla v(x) = 0$

(Uraltseva: $p > 2$, Lewis, DiBenedetto: $1 < p < 2$),

- a continuous v is called a viscosity solution if:



$\nabla\phi(x_0) \neq 0$
$\Rightarrow \Delta_p \phi(x_0) \leq 0$



$\nabla\phi(x_0) \neq 0$
$\Rightarrow \Delta_p \phi(x_0) \geq 0$

- weak solutions = viscosity solutions (Juutinen-Lindqvist-Manfredi 2001)
- Open problems: If $v_{|B_\rho} = 0 \overset{?}{\Rightarrow} v \equiv 0$ in Ω

  (true for $n = 2$ by complex analysis methods: Manfredi 1988)

Proof of Thm 1 ($\beta \neq 0$): existence of p-harmonious u

(after Luiro, Parviainen, Saksman 2013)

Let $u : X \to \mathbb{R}$ bounded, Borel. Look for fixed points of the operator:

$$(Tu)(x) = \begin{cases} \frac{\alpha}{2} \sup_{B_\varepsilon(x)} u + \frac{\alpha}{2} \inf_{B_\varepsilon(x)} u + \beta \fint_{B_\varepsilon(x)} u & \text{in } \Omega \\ u(x) & \text{in } \Gamma_\varepsilon \end{cases}$$

- Tu is also bounded, Borel.
- Let $u_0 = \begin{cases} \inf_{\Gamma_\varepsilon} F & \text{in } \Omega \\ F & \text{in } \Gamma_\varepsilon \end{cases}$
- Then $u_1 = Tu_0 \geq u_0$ and $u_2 = Tu_1 \geq Tu_0 = u_1$ and ... $u_{j+1} \geq u_j$

Hence $u_j \nearrow u$. In fact $u_j \rightrightarrows u$ in Ω.

- So: $u \leftleftarrows u_{j+1} = Tu_j \rightrightarrows Tu$, so $Tu = u$ and $u = F$ on Γ_ε.

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Proof of Thm 1 ($\beta \neq 0$): existence of p-harmonious u

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- So: $u \leftleftarrows u_{j+1} = Tu_j \rightrightarrows Tu$, so $Tu = u$ and $u = F$ on Γ_ε.
- Uniqueness: If $u \neq v$ are two solutions, call $M = \sup(u - v) > 0$.

$$\forall x \quad u(x) - v(x) = \frac{\alpha}{2}(\sup_{B_\varepsilon(x)} u - \sup_{B_\varepsilon(x)} v) + \frac{\alpha}{2}(\inf_{B_\varepsilon(x)} u - \inf_{B_\varepsilon(x)} v)$$
$$+ \beta \fint_{B_\varepsilon(x)} (u - v) \leq \alpha \sup_{B_\varepsilon(x)} (u - v) + \beta \fint_{B_\varepsilon(x)} (u - v)$$
$$\leq \alpha M + \beta \fint_{B_\varepsilon(x)} (u - v)$$

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Proof of Thm 1 ($\beta \neq 0$): existence of p-harmonious u

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Let $u : X \to \mathbb{R}$ bounded, Borel. Look for fixed points of the operator:

$$(Tu)(x) = \begin{cases} \frac{\alpha}{2} \sup_{B_\varepsilon(x)} u + \frac{\alpha}{2} \inf_{B_\varepsilon(x)} u + \beta ⨍_{B_\varepsilon(x)} u & \text{in } \Omega \\ u(x) & \text{in } \Gamma_\varepsilon \end{cases}$$

- Tu is also bounded, Borel.
- Let $u_0 = \begin{cases} \inf_{\Gamma_\varepsilon} F & \text{in } \Omega \\ F & \text{in } \Gamma_\varepsilon \end{cases}$
- Then $u_1 = Tu_0 \geq u_0$ and $u_2 = Tu_1 \geq Tu_0 = u_1$ and ... $u_{j+1} \geq u_j$

Hence $u_j \nearrow u$. In fact $u_j \rightrightarrows u$ in Ω.

- So: $u \leftleftarrows u_{j+1} = Tu_j \rightrightarrows Tu$, so $Tu = u$ and $u = F$ on Γ_ε.
- Uniqueness: If $u \neq v$ are two solutions, call $M = \sup(u - v) > 0$.

$\forall x \quad u(x) - v(x) \leq \alpha M + \beta ⨍_{B_\varepsilon(x)} (u - v)$.

If $u(x_n) - v(x_n) \to M$ and $x_n \to x_0 \in X$ then: $M \leq \alpha M + \beta ⨍_{B_\varepsilon(x_0)} (u - v)$.

But $\beta = 1 - \alpha$, so: $M \leq ⨍_{B_\varepsilon(x_0)} (u - v)$. Hence $u - v \equiv M$ in $B_\varepsilon(x_0)$.

By extension: $u - v \equiv M$ in X, contradicting $u = v$ on Γ_ε. ∎

Proof of Thm 1: u is the game value

Lemma Let $u : \Omega \to \mathbb{R}$ be bounded and Borel. Fix $\varepsilon, \delta > 0$.

There exists suboptimal selections $S_{sup}, S_{inf} : \Omega \to \Omega$, such that:

- S_{sup}, S_{inf} are Borel functions, and $\forall x \in \Omega\ \ S_{sup}(x), S_{inf}(x) \in B_\varepsilon(x)$
- $\forall x \in \Omega\ \ u(S_{sup}(x)) \geq \sup_{B_\varepsilon(x)} u - \delta, \ \ u(S_{inf}(x)) \leq \inf_{B_\varepsilon(x)} u + \delta$

Example S_{sup} may not exist with closed balls $\bar{B}_\varepsilon(x)$!

Let $A \subset \mathbb{R}^3$ be bounded, Borel, and such that: $A + \bar{B}_1(0)$ is not Borel.

Let $u = \chi_A$. Assume that: $\forall x\ \ u(S_{sup}(x)) \geq \sup_{\bar{B}_1(x)} u - \frac{1}{2}$.

Then: $(S_{sup})^{-1}(A) = \{x \in \mathbb{R}^3;\ \sup_{\bar{B}_1(x)} u = 1\} = A + \bar{B}_1(0)$

Therefore S_{sup} cannot be a Borel function.

Proof of Thm 1: u is the game value

Recall: $u_I(x_0) = \sup_{\sigma_I} \inf_{\sigma_{II}} \mathbb{E}^{x_0}_{\sigma_I,\sigma_{II}}[F_\tau]$,

$u_{II}(x_0) = \inf_{\sigma_{II}} \sup_{\sigma_I} \mathbb{E}^{x_0}_{\sigma_I,\sigma_{II}}[F_\tau]$.

Clearly: $u_I \leq u_{II}$.

- We show that $u_{II} \leq u$. Symetrically $u \leq u_I$. Then $u_I = u_{II} = u$.
- Fix $\eta > 0$. Choose $\bar{\sigma}_{II}$ suboptimal selection, so that:

$$u(\bar{\sigma}^k_{II}(x_0 \ldots x_k)) \leq \inf_{B_\varepsilon(x_k)} u + \frac{\eta}{2^{k+1}}$$

- Let σ_I be any strategy for player I.

$$u_{II}(x_0) \leq \sup_{\sigma_I} \mathbb{E}^{x_0}_{\sigma_I,\bar{\sigma}_{II}}[F_\tau] \leq \sup_{\sigma_I} \mathbb{E}^{x_0}_{\sigma_I,\bar{\sigma}_{II}}\left[u(x_\tau) + \frac{\eta}{2^\tau}\right]$$

$$\leq \sup_{\sigma_I} \mathbb{E}^{x_0}_{\sigma_I,\bar{\sigma}_{II}}\left[u(x_0) + \frac{\eta}{2^0}\right] = u(x_0) + \eta$$

$$\Uparrow$$

because $\left\{u(x_k) + \frac{\eta}{2^k},\ (\text{Borel } \sigma\text{-algebra})^k\right\}$ is a supermartingale.

Proof of Thm 1: u is the game value

Lemma $\left\{u(x_k)+\frac{\eta}{2^k},\ \mathcal{F}_k^{x_0}\right\}$ is a supermartingale.

Proof $\mathbb{E}^{x_0}_{\sigma_I,\bar\sigma_{II}}\left\{u(x_k)+\frac{\eta}{2^k} \mid \mathcal{F}^{x_0}_{k-1}\right\}(x_0\ldots x_{k-1}) =$

$$= \int_X \left(u+\frac{\eta}{2^k}\right)\,\mathrm{d}\gamma_{k-1}[x_0\ldots x_{k-1}]$$

$$= \frac{\alpha}{2}\,u(\sigma_I(x_0\ldots x_{k-1})) + \frac{\alpha}{2}\,u(\bar\sigma_{II}(x_0\ldots x_{k-1})) + \beta \fint_{B_\varepsilon(x_{k-1})} u\,\mathrm{d}y + \frac{\eta}{2^k}$$

$$\le \frac{\alpha}{2}\,\sup_{B_\varepsilon(x_{k-1})} u + \frac{\alpha}{2}\left(\inf_{B_\varepsilon(x_{k-1})} u + \frac{\eta}{2^k}\right) + \beta \fint_{B_\varepsilon(x_{k-1})} u\,\mathrm{d}y + \frac{\eta}{2^k}$$

$$= u(x_{k-1}) + \left(\frac{\alpha}{2}+1\right)\frac{\eta}{2^k} \le u(x_{k-1}) + \frac{\eta}{2^{k-1}} \qquad \blacksquare$$

Theorem (Doob's optional stopping)

If $\{f_k,\mathcal{F}_k\}$ is a supermartingale and τ_1,τ_2 are bounded stopping times,

then: $\boxed{\tau_1\le\tau_2 \ \Rightarrow\ \mathbb{E}[f_{\tau_2}]\le\mathbb{E}[f_{\tau_1}].}$

p-harmonious u and the martingale calculation

Similarly, one can prove:

Lemma Let $\bar\sigma_I, \bar\sigma_{II}$ be any two fixed strategies. Then:

$$\inf_{\sigma_{II}} \mathbb{E}^{x_0}_{\bar\sigma_I,\sigma_{II}}[u(x_{\tau^*})] \le u(x_0) \le \sup_{\sigma_I} \mathbb{E}^{x_0}_{\sigma_I,\bar\sigma_{II}}[u(x_{\tau^*})]$$

for every stopping time $\tau^* \le \tau$.

Proof Use the suboptimal selection strategy $\tilde\sigma_{II}$:

$$\begin{aligned}\inf_{\sigma_{II}} \mathbb{E}^{x_0}_{\bar\sigma_I,\sigma_{II}}[u(x_{\tau^*})] &\le \mathbb{E}^{x_0}_{\bar\sigma_I,\tilde\sigma_{II}}[u(x_{\tau^*})] \le \mathbb{E}^{x_0}_{\bar\sigma_I,\tilde\sigma_{II}}\left[u(x_{\tau^*}) + \frac{\eta}{2^{\tau_*}}\right] \\ &\le \mathbb{E}^{x_0}_{\bar\sigma_I,\tilde\sigma_{II}}\left[u(x_0) + \frac{\eta}{2^0}\right] = u(x_0) + \eta\end{aligned}$$

because $\left\{u(x_k) + \frac{\eta}{2^k},\ \mathcal{F}^{x_0}_k\right\}$ is a supermartingale. ■

Proof of Thm 2: $u_\varepsilon \rightrightarrows v$ continuous

(after Manfredi, Parviainen, Rossi 2012)

Lemma ($\sim$ Ascoli-Arzela theorem for discontinuous functions)

Let $u_\varepsilon : \bar{\Omega} \to \mathbb{R}$ be a sequence of functions such that:

(i) $\exists C \ \ \forall x \in \bar{\Omega} \ \ \forall \varepsilon \quad |u_\varepsilon(x)| \leq C$

(ii) $\forall \eta > 0 \ \ \exists r_0 \ \ \exists \varepsilon_0 \ \ \forall_{\substack{x,y \in \bar{\Omega} \\ |x-y| < r_0}} \quad \forall \varepsilon < \varepsilon_0 \quad |u_\varepsilon(x) - u_\varepsilon(y)| < \eta$

Then $u_\varepsilon \rightrightarrows v$ in $\bar{\Omega}$ and v is a continuous function. $\blacksquare$

- Take $u_\varepsilon : \Omega \cup \Gamma_\varepsilon \to \mathbb{R}$ to be the unique p-harmonious functions

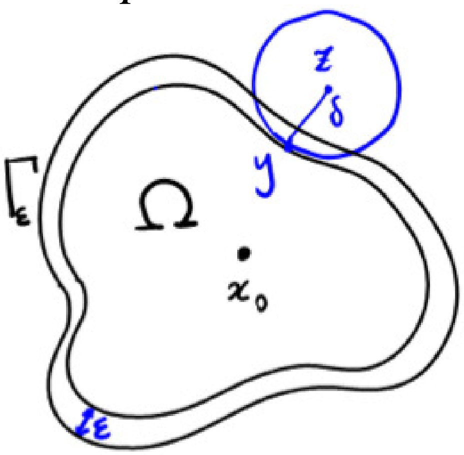
with parameter ε and boundary data F in Γ_ε

- u_ε is equibounded. Need to verify condition (ii).
- Case 1: $x \in \Omega, \ y \in \partial\Omega \quad \Rightarrow \quad$ Case 2: $x, y \in \Omega$.

Proof of Thm 2: verification of (ii) for $x_0 \in \Omega,\ y \in \partial\Omega$

(after Manfredi, Parviainen, Rossi 2012)

$$|u_\varepsilon(x_0) - u_\varepsilon(y)| = |u_\varepsilon(x_0) - F(y)| \le |u_\varepsilon(x_0) - F(z)| + L\delta$$
$$= |\sup_{\sigma_I} \inf_{\sigma_{II}} \mathbb{E}^{x_0}_{\sigma_I,\sigma_{II}}[F(x_\tau) - F(z)]| + L\delta \le L \sup_{\sigma_I} \inf_{\sigma_{II}} \mathbb{E}^{x_0}_{\sigma_I,\sigma_{II}}[|x_\tau - z|] + L\delta$$



Take any σ_I and $\sigma_{II} = \bar{\sigma}^z_{II}$ "pulling towards z"

$$\bar{\sigma}^z_{II}(x_k) = \begin{cases} x_k + \varepsilon \frac{z - x_k}{|z - x_k|} & \text{if } x_k \notin \bar{B}_\delta(z) \\ x_k & \text{if } x_k \in \bar{B}_\delta(z) \end{cases}$$

Compute: $\mathbb{E}^{x_0}_{\sigma_I,\bar{\sigma}^z_{II}} \left\{ |x_k - z| - C\varepsilon^2 k \mid \mathcal{F}^{x_0}_{k-1} \right\} (x_0 \ldots x_{k-1})$

$$\le \frac{\alpha}{2}(|x_{k-1} - z| + \varepsilon) + \frac{\alpha}{2}(|x_{k-1} - z| - \varepsilon) + \beta \fint_{B_\varepsilon(x_{k-1})} |s - z| \, \mathrm{d}s - C\varepsilon^2 k$$
$$= \alpha |x_{k-1} - z| + \beta \left(|x_{k-1} - z| + \frac{\varepsilon^2}{8} \Delta(|s - z|) + o(\varepsilon^2) \right) - C\varepsilon^2 k$$
$$\le |x_{k-1} - z| - C\varepsilon^2 (k-1) \quad \text{for } C >> 1.$$

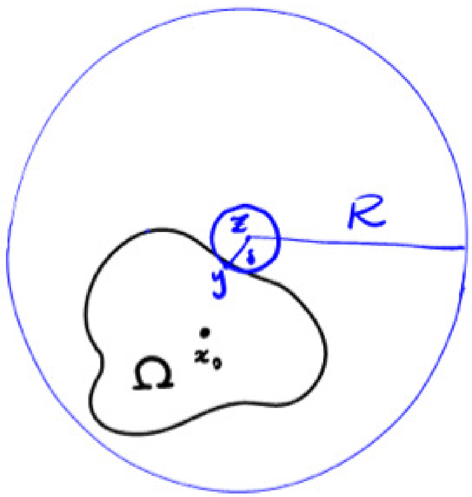
Hence: $\left\{ |x_k - z| - C\varepsilon^2 k, \mathcal{F}^{x_0}_k \right\}$ is a supermartingale.

Proof of Thm 2: verification of (ii) for $x_0 \in \Omega,\ y \in \partial\Omega$

(after Manfredi, Parviainen, Rossi 2012)

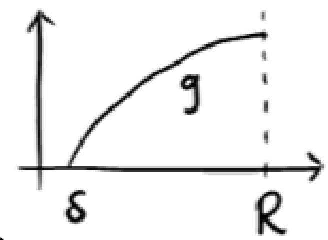
By Doob's Thm: $\mathbb{E}^{x_0}_{\sigma_I,\bar\sigma^z_{II}}\left[|x_\tau - z| - C\varepsilon^2\tau\right] \leq \mathbb{E}^{x_0}_{\sigma_I,\bar\sigma^z_{II}}\left[|x_0 - z|\right] = |x_0 - z|$

Hence: $\mathbb{E}^{x_0}_{\sigma_I,\bar\sigma^z_{II}}\left[|x_\tau - z|\right] \leq |x_0 - z| + C\varepsilon^2\mathbb{E}^{x_0}_{\sigma_I,\bar\sigma^z_{II}}[\tau]$



Consider:

$$\begin{cases} \Delta g = -2(n+2) & \text{in } B_R(z)\setminus \bar B_\delta(z) \\ g = 0 & \text{on } \partial B_\delta(z) \\ \frac{\partial g}{\partial \vec n} = 0 & \text{on } \partial B_R(z) \end{cases}$$



i.e: $g(s) = g(|s-z|)$ and $g(r) = -ar^2 - b\log r + c$

g satisfies: $\forall B_\varepsilon(s_0) \subset$ annulus $\fint_{B_\varepsilon(s_0)} g(s)\,\mathrm{d}s = g(s_0) - \varepsilon^2$

- Compute: $\mathbb{E}^{x_0}_{\sigma_I,\bar\sigma^z_{II}}\left\{g(|x_k - z|) + C\varepsilon^2 k \mid \mathcal{F}^{x_0}_{k-1}\right\}(x_0 \dots x_{k-1})$

$$\leq \frac{\alpha}{2}g(|x_{k-1}-z|+\varepsilon) + \frac{\alpha}{2}g(|x_{k-1}-z|-\varepsilon) + \beta\left(g(|x_{k-1}-z|) - \varepsilon^2\right) + C\varepsilon^2 k$$

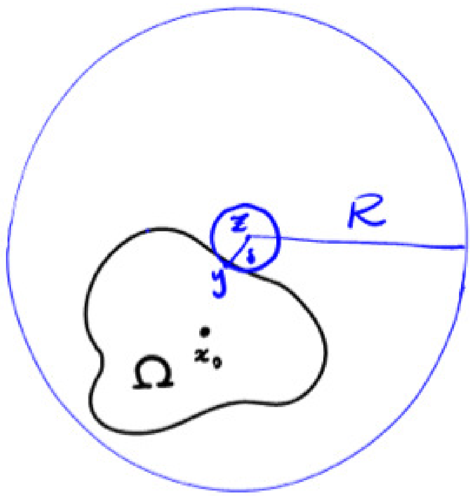
$\leq g(|x_{k-1}-z|) + C\varepsilon^2(k-1)$ because g is increasing and concave.

Proof of Thm 2: verification of (ii) for $x_0 \in \Omega,\ y \in \partial\Omega$

(after Manfredi, Parviainen, Rossi 2012)

Hence, Doob's Thm yields: $\mathbb{E}^{x_0}_{\sigma_I, \bar{\sigma}^z_{II}}[g(|x_{\tau^*} - z|) + C\varepsilon^2\tau^*] \le g(|x_0 - z|)$

where τ^* is the exit time into the inner ball $\bar{B}_\delta(z)$. Clearly: $\tau \le \tau^*$



Hence:

$$\mathbb{E}^{x_0}_{\sigma_I, \bar{\sigma}^z_{II}}[C\varepsilon^2\tau] \le \mathbb{E}^{x_0}_{\sigma_I, \bar{\sigma}^z_{II}}[C\varepsilon^2\tau^*]$$
$$\le g(|x_0 - z|) - \mathbb{E}^{x_0}_{\sigma_I, \bar{\sigma}^z_{II}}[g(|x_{\tau^*} - z|)]$$
$$\le L_g \operatorname{dist}(x_0, \partial B_\delta(z)) + L_g\varepsilon \le L_g(|x_0 - z| + \varepsilon)$$

Recall: $\boxed{\mathbb{E}^{x_0}_{\sigma_I, \bar{\sigma}^z_{II}}[|x_\tau - z|] \le |x_0 - z| + C\varepsilon^2 \mathbb{E}^{x_0}_{\sigma_I, \bar{\sigma}^z_{II}}[\tau]} \le C_\delta(|x_0 - z| + \varepsilon)$

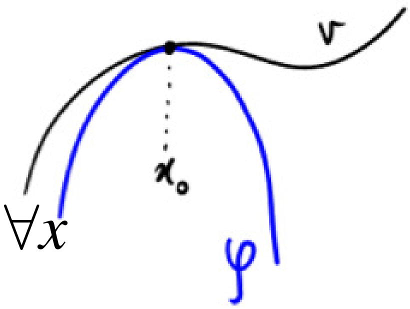
- Concluding: $|u_\varepsilon(x_0) - u_\varepsilon(y)| \le C_{\delta,\Omega}(|x_0 - y| + \varepsilon) + C_F\delta$

which gives exactly condition (ii)! ■

Proof of Thm 2: the limit satisfies $\Delta_p v = 0$

(after Manfredi, Parviainen, Rossi 2012)

We proved that: $u_\varepsilon \rightrightarrows v$ in $\bar{\Omega}$. Clearly $v = F$ on $\partial\Omega$



Verification that v is a viscosity solution:

Find $x_\varepsilon \to x_0$ so that: $(u_\varepsilon - \phi)(x_\varepsilon) \leq (u_\varepsilon - \phi)(x) + \varepsilon^3 \ \forall x$

- $u_\varepsilon(x_\varepsilon) = \frac{\alpha}{2} \sup_{B_\varepsilon(x_\varepsilon)} u_\varepsilon + \frac{\alpha}{2} \inf_{B_\varepsilon(x_\varepsilon)} u_\varepsilon + \beta \fint_{B_\varepsilon(x_\varepsilon)} u_\varepsilon$
- $\phi(x_\varepsilon) = \frac{\alpha}{2} \sup_{B_\varepsilon(x_\varepsilon)} \phi + \frac{\alpha}{2} \inf_{B_\varepsilon(x_\varepsilon)} \phi + \beta \fint_{B_\varepsilon(x_\varepsilon)} \phi$
$$-\frac{\beta\varepsilon^2}{8}\big(\Delta\phi(x_\varepsilon) + (p-2)\Delta_\infty\phi(u_\varepsilon)\big) + o(\varepsilon^2)$$

Hence: $(u_\varepsilon - \phi)(x_\varepsilon) = \frac{\alpha}{2}(\sup u_\varepsilon - \sup \phi) + \frac{\alpha}{2}(\inf u_\varepsilon - \inf \phi) + \beta \fint (u_\varepsilon - \phi)$
$$+ \frac{\beta\varepsilon^2}{8} \frac{1}{|\nabla\phi|^{p-2}} \Delta_p \phi(x_\varepsilon) + o(\varepsilon^2)$$
$$\geq (u_\varepsilon - \phi)(x_\varepsilon) + \frac{\beta\varepsilon^2}{8} \frac{1}{|\nabla\phi|^{p-2}} \Delta_p \phi(x_\varepsilon) + o(\varepsilon^2).$$

Consequently: $\frac{1}{|\nabla\phi|^{p-2}} \Delta_p \phi(x_\varepsilon) \leq o(1)$.

Since $\nabla\phi(x_0) \neq 0$, we get: $\Delta_p \phi(x_0) \leq 0$. ∎

Harnack's inequality for p-harmonious functions ($2 \leq p < \infty$)

(after Luiro, Parviainen, Saksman 2013)

THM 3 For each ε, let u_ε be a p-harmonious function in Ω. Then:

$\forall B_\rho(z_0) \subset B_r(z_0)$ $\exists C_p$ $\forall \varepsilon < \rho$ $\text{osc}(u_\varepsilon, B_\rho) \leq C_p \frac{\rho}{r} \text{osc}(u_\varepsilon, B_r)$
$B_{2r}(z_0) \subset \Omega$

Corollary Let u_ε be a positive p-harmonious function in $B_{2r}(z_0)$.

Then there exists C (independent of ε) so that:

$$\forall B_\rho(x_0) \subset B_r(z_0) \quad \sup_{B_\rho} u_\varepsilon \leq C \inf_{B_\rho} u_\varepsilon$$

Corollary Harnack's inequality for p-harmonic v: $\sup_{B_\rho} v \leq C \inf_{B_\rho} v$

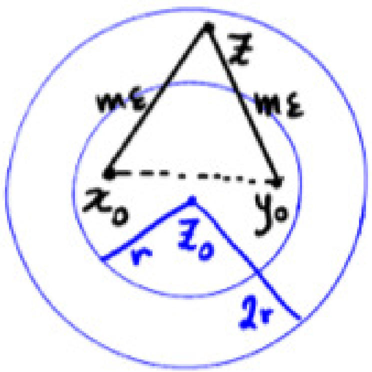
Prior proofs of Harnack's inequality:

- Ladyzhenskaya-Uraltseva 1968, Serrin 1967
- Nash-Moser iteration, DeGiorgi classes (DiBenedetto-Trudinger 1984)
- Here: completely different proof!

Proof of Thm 3: Harnack's inequality

(after Luiro, Parviainen, Saksman 2013)

Let $x_0, y_0 \in B_r(z_0)$. Let $z \in B_{2r}$: $|x_0 - z| = m\varepsilon = |y_0 - z|$, $m = \lfloor \frac{|x_0 - y_0|}{\varepsilon} \rfloor$.



Define the strategy $\bar{\sigma}^z_{II}$:

- player II cancels the earliest move of player I
- otherwise player II "pulls towards z" (by ε)

Position of token:

$$x_{k+1} = x_0 + \sum_{i \in J^k_I} w_i + \sum_{i \in J^k_{II}} w_i + \sum_{i \in J^k_{random}} w_i$$

Define the stopping time $\tau^*_{II} = \min\{k;\ (i), (ii), (iii)\}$:

(i) player II has won m times more than player I: $|J^k_{II}| = |J^k_I| + m$

(ii) player I has won $\lceil \frac{2r}{\varepsilon} \rceil$ times more than player II: $|J^k_I| = |J^k_{II}| + \lceil \frac{2r}{\varepsilon} \rceil$

(iii) total random moves are large: $|\sum_{i \in J^k_{random}} w_i| \geq 2r$.

In particular: $x_{\tau^*_{II}} \in B_{6r}(z_0)$. Also: $\tau^*_{II} \leq \tau$, so (by lemma):

$$\boxed{u(x_0) - u(y_0) \leq \sup_{\sigma_I, \sigma_{II}} \left(\mathbb{E}^{x_0}_{\sigma_I, \bar{\sigma}^z_{II}} [u_{\tau^*_{II}}] - \mathbb{E}^{y_0}_{\bar{\sigma}^z_I, \sigma_{II}} [u_{\tau^*_I}] \right)}$$

Proof of Thm 3: Harnack's inequality

(after Luiro, Parviainen, Saksman 2013)

Fix σ_I, σ_{II}. <u>Then:</u> $\boxed{\mathbb{E}^{x_0}_{\sigma_I,\bar\sigma^z_{II}}[u_{\tau^*_{II}}] - \mathbb{E}^{y_0}_{\bar\sigma^z_I,\sigma_{II}}[u_{\tau^*_I}]}$

$$= \mathbb{E}^{x_0}_{\sigma_I,\bar\sigma^z_{II}}[u_{\tau^*_{II}}\,\chi_{\text{game ends by (i)}}] + \mathbb{E}^{x_0}_{\sigma_I,\bar\sigma^z_{II}}[u_{\tau^*_{II}}\,\chi_{\text{game ends by (ii) or (iii)}}]$$

$$- \mathbb{E}^{y_0}_{\bar\sigma^z_I,\sigma_{II}}[u_{\tau^*_I}\,\chi_{\text{game ends by (i)}}] - \mathbb{E}^{y_0}_{\bar\sigma^z_I,\sigma_{II}}[u_{\tau^*_I}\,\chi_{\text{game ends by (ii) or (iii)}}]$$

$$= \mathbb{E}^{x_0}_{\sigma_I,\bar\sigma^z_{II}}[u_{\tau^*_{II}}\,\chi_{\text{game ends by (ii) or (iii)}}] - \mathbb{E}^{y_0}_{\bar\sigma^z_I,\sigma_{II}}[u_{\tau^*_I}\,\chi_{\text{game ends by (ii) or (iii)}}]$$

$$\le P\,\Big(\sup_{B_{6r}(z_0)} u - \inf_{B_{6r}(z_0)} u\Big) \le \boxed{C_p\frac{|x_0-y_0|}{r}\Big(\sup_{B_{6r}(z_0)} u - \inf_{B_{6r}(z_0)} u\Big)}$$

where $P =$ probability that game ends by (ii) or (iii).

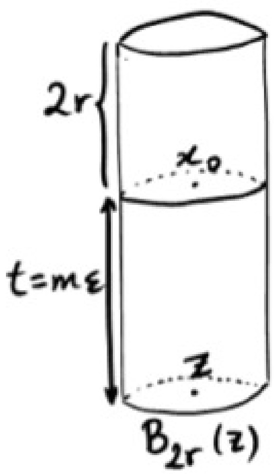
- Concluding: $|u(x_0)-u(y_0)| \le C_p\frac{|x_0-y_0|}{r}\Big(\sup_{B_{6r}(z_0)} u - \inf_{B_{6r}(z_0)} u\Big)$

$$\Rightarrow \quad \operatorname{osc}(u,B_\rho) \le C_p\,\frac{\rho}{6r}\,\Big(\sup_{B_{6r}} u - \inf_{B_{6r}} u\Big) \qquad \blacksquare$$

Proof of Thm 3: $P \le C_p \frac{|x_0 - y_0|}{r}$

(after Luiro, Parviainen, Saksman 2013)

To estimate P, couple the game with a cylinder walk:



Rules of the walk:

- token initially at $(z, m\varepsilon)$
- probability $\frac{\alpha}{2} \to$ player I moves by $(0, \varepsilon)$
- probability $\frac{\alpha}{2} \to$ player II moves by $(0, -\varepsilon)$
- probability $\beta \to$ random move by $(y, 0)$, $y \in B_\varepsilon(0)$

Lemma $P =$ probability (the walk does not exit the cylinder through its bottom)

$$\le C_p \frac{m\varepsilon + \varepsilon}{2r}$$

■

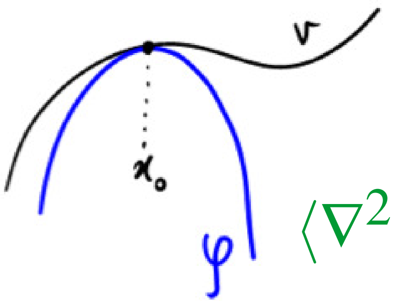
The limiting case $p \to \infty$ $(\beta = 0)$

Theorem (Bhattacharya-DiBenedetto-Manfredi 1989, Jensen 1993)

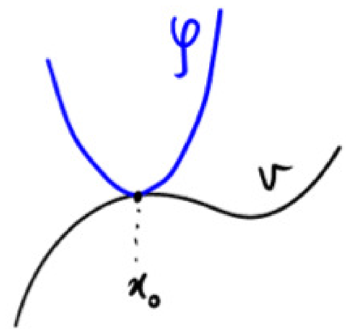
Let v_p be the unique solution to: $\Delta_p v_p = 0, \; v_{p|\partial\Omega} = F.$

Then $v_p \rightrightarrows v_\infty$ in $\bar{\Omega}$ as $p \to \infty$, and v_∞ is the unique viscosity solution to:

$$\boxed{\Delta_\infty v_\infty = 0, \; v_{\infty|\partial\Omega} = F}$$



$$\nabla\phi(x_0) \neq 0$$
$$\Rightarrow$$
$$\langle \nabla^2\phi : \frac{\nabla\phi}{|\nabla\phi|} \otimes \frac{\nabla\phi}{|\nabla\phi|} \rangle (x_0) \leq 0$$



$$\nabla\phi(x_0) \neq 0$$
$$\Rightarrow \Delta_\infty \phi(x_0) \geq 0$$

Theorem (Aronsson, Aronsson-Crandall-Juutinen 1967, 2004)

$v_\infty \in \mathrm{Lip}(\bar{\Omega}, \mathbb{R})$ has the uniquely defining property:

$$\forall \text{ open } V \subset \Omega \;\; \forall v : V \to \mathbb{R}, \; v_{|\partial V} = v_{\infty|\partial V} \qquad \mathrm{Lip}(v_\infty, V) \leq \mathrm{Lip}(v, V)$$

- v_∞ is AMLE (Absolutely Minimizing Lipschitz Extension)

" $\Delta_\infty v = 0$ is the Euler-Lagrange equation of $\|\nabla v\|_\infty$"

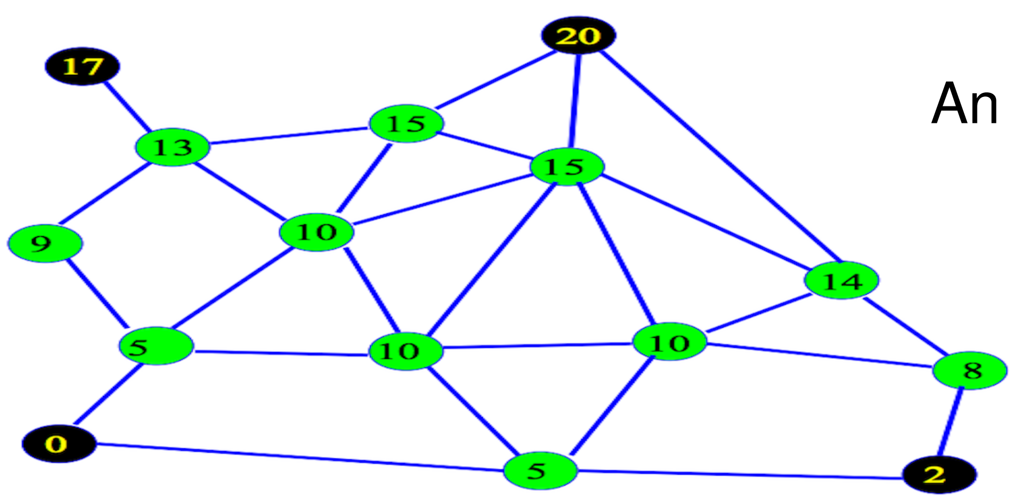
The limiting case $p = \infty$

- $\Delta_\infty v = 0$ is a fully nonlinear equation,
  only viscosity solution definition valid (no integration by parts)
- $n = 2$: $v_\infty \in C^{1,\alpha}$ (Savin, Savin-Evans 2008),
  $n > 2$: v_∞ differentiable everywhere (Evans-Smart 2011)

<u>Thm</u> (Peres-Schramm-Sheffield-Wilson 2008)

$u_\varepsilon \rightrightarrows v_\infty$ as $\varepsilon \to 0$ where u_ε are values of the Tug-of-War game:

$$\begin{cases} u_\varepsilon(x) = \frac{1}{2}\sup_{B_\varepsilon(x)} u_\varepsilon + \frac{1}{2}\inf_{B_\varepsilon(x)} u_\varepsilon & \text{in } \Omega \\ u_\varepsilon = F & \text{on } \Gamma_\varepsilon \end{cases}$$



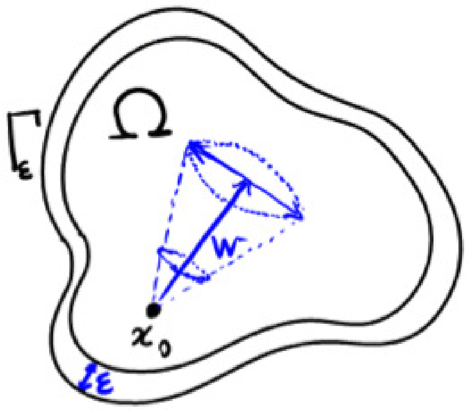
An AMLE extension

is inifinity harmonic!

The limiting case $p \to 1$

Thm (Peres-Sheffield 2008)

Let $1 < p < \infty$. Then the values u_ε of the following game converge, as $\varepsilon \to 0$ to v with: $\Delta_p v = 0, \ \ v_{|\partial\Omega} = F$.



- token initially at $x_0 \in \Omega$.
- the winning player (probability $\frac{1}{2}$) moves token by vector w ($|w| < \varepsilon$)
- token moved randomly by $(p-1)^{-1/2}|w|$ in the direction orthogonal to w

Game stops when token position in Γ_ε. Player II pays $F(x_n)$ to player I.

Observe that the following structure is needed to define the game:

- $p = \infty$ metric (extensions to length spaces by PSSW)
- $2 \le p < \infty$ measure (extensions to Heisenberg groups (Carnot-Caratheodory groups) by Ferrari-Liu-Manfredi)
- $1 < p < 2$ perpendicularity: Riemannian structure (other extensions: Barron-Evans-Jenssen)

The limiting case $p \to 1$

Thm (Juutinen 2005, Sternberg-Williams-Ziemer 1992)

Let Ω be convex. Let v_p be unique solutions to: $\Delta_p v_p = 0,\ v_{p|\partial\Omega} = F$.

Then $v_p \rightrightarrows v_1$ in Ω as $p \to 1$, and v_1 is the function of least gradient:

$$v_1 \in BV(\Omega) \cap C(\Omega) \text{ and } v_1 \text{ minimizes } \|\nabla v\|_{BV(\Omega)}$$

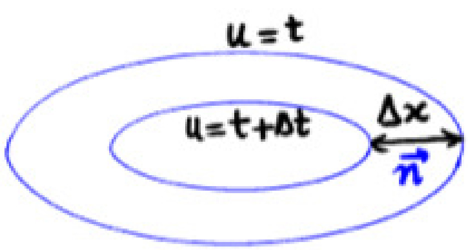
Moreover, v_1 is a viscosity solution to Δ_1:

$$\boxed{\Delta_1 v_1 = \text{div}\left(\frac{\nabla v_1}{|\nabla v_1|}\right) = 0, \quad v_{1|\partial\Omega} = F}$$

However, there exist viscosity solutions to Δ_1, other than v_1.

Motion by curvature and Δ_1

Level set $v = t$ is the image of $\partial\Omega$ under motion by curvature for time t

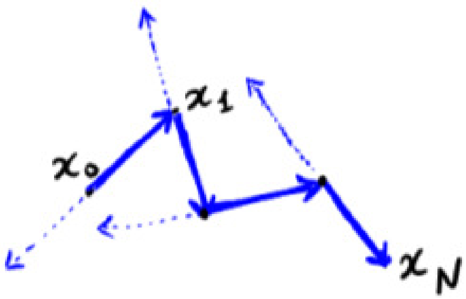


- curvature of the level set: $\kappa = -\text{div}\left(\frac{\nabla v}{|\nabla v|}\right)$
- $|\nabla v| = |\nabla v \cdot \vec{n}| = \frac{\partial v}{\partial \vec{n}} \approx \frac{\Delta t}{\Delta x}$
- $\kappa = \frac{\Delta x}{\Delta t} \quad \Leftrightarrow \quad -\Delta_1 v = \frac{1}{|\nabla v|}$

$$\boxed{|\nabla v| \Delta_1 v + 1 = 0, \quad v_{|\partial\Omega} = 0}$$

Motion by mean curvature

The Paul/Carol game in $\mathbb{R}^2$:



- token initially at $x_0 \in \mathbb{R}^2$
- given $t_0 > 0$, the length of game $N = \lfloor \frac{t_0}{\varepsilon^2} \rfloor$ steps
- player I (Paul) chooses w_k, $|w_k| = 1$
- player II (Carol) chooses $b_k = \pm 1$
- token moved to $x_{k+1} = x_k + \sqrt{2}\varepsilon\, b_k w_k$

After N steps, Paul pays $u_0(x_N)$ to Carol.

Game value:

$$\begin{cases} u_\varepsilon(x_0, t_0) = \min\limits_{w_1} \max\limits_{b_1} \ldots \min\limits_{w_N} \max\limits_{b_N} u_0(x_N) \\ u_\varepsilon(x_0, 0) = u_0(x_0) \end{cases}$$

Thm (Kohn-Serfaty 2006) Let $u_0 \in C_c(\mathbb{R}^2)$. Then $u_\varepsilon \rightrightarrows v$ in $\mathbb{R}^2 \times [0, \infty)$ locally uniformly as $\varepsilon \to 0$, and v is the unique viscosity solution of:

$$\begin{cases} v_t - |\nabla v| \Delta_1 v = 0 & \text{in } \mathbb{R}^2 \times (0, \infty) \\ v(x, 0) = u_0(x) & \text{in } \mathbb{R}^2 \end{cases}$$

- existence/uniqueness: Evans-Spruck, Chen-Giga-Goto 1991.

Conclusions

- $\exists$ connection between: potential theory/nonlinear PDEs, and probability theory/ stochastic tug-of-war games
- based on: harmonic/$p-$harmonic functions and martingales share a common cancellation property, via mean value property
- variety of PDEs and other problems:
  – Harnack inequality (Luiro-Parviainen-Saksman)
  – continuous u_ε, finite difference scheme (Armstrong-Smart)
  – obstacle problem (Manfredi-Rossi-Somersille, L-Manfredi)
  – Neumann boundary condition (Antunovic-Peres-Sheffield-Somersille)
- in fact: $\exists$ game interpretation for a large class of elliptic and parabolic, nonlinear equations (Kohn-Serfaty 2010)
- advantage: direct proofs by assigning “strategies” $\Rightarrow$ “inequalities”. Probability tools used for nonlinear equations.

THANK YOU.