A generalization of the Hopf-Cole transformation for stationary Mean Field Games systems

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1. Mean Field Games

- Is a branch of dynamic games which aims at modeling and analyzing decision processes involving a very large number of indistinguishable rational agents.
- Proposed independently by Lasry, Lions and Caines, Huang, Malhamé ($\sim$2005).
- Macroscopic point of view: characterize the optimal (in the sense of Nash) distribution of the entire population.

The framework

Every player move in the state space $\Omega \subseteq \mathbb{R}^d$ according to

$$dX_t^{(i)} = -\alpha_t^{(i)} dt + \sqrt{2} dB_t^{(i)} \quad \in \Omega$$

and pays a cost of the form

$$\mathcal{J}(X_0^{(i)}, \alpha^{(i)}) = \liminf_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}[L(\alpha_s^{(i)}) + f(X_s^{(i)}, \hat{m}_s)] ds,$$

where $\hat{m}_t := \frac{1}{N-1} \sum_{j \neq i} dX_t^{(j)}$, players are indistinguishable!.

Every player aims at minimizing his own cost.

It has been shown that ([Lasry, Lions] and [Huang, Caines, Malhame], '05)
Nash equilibria of the game with a large number of players can be approximated as $N \to \infty$ by the stationary Mean Field Games system

$$
\begin{cases}
\text{(HJB)} & -\Delta u + H(Du) + \lambda = f(x, m), \quad \text{in } \Omega, \\
\text{(K)} & -\Delta m - \operatorname{div}(DH(Du)\, m) = 0, \\
 & \int_\Omega m = 1,
\end{cases}
$$

where $H(p)$ is the Legendre transform of $L(\alpha)$,

- m is the equilibrium probability distribution,
- $\alpha^* : x \mapsto -DH(Du(x))$ provides the optimal (ϵ)-Nash strategy in feedback form,
- λ is the average cost.

Interpretation of MFG (1)

The MFG system can be intended as an optimality condition for the following control problem:

A "typical" player

$$dX_t = -\alpha_t dt + \sqrt{2} dB_t \quad \in \Omega$$

aims at minimizing

$$\mathcal{J}(X_0^{(i)}, \alpha) = \liminf_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}[L(\alpha_s) + f(X_s, m(X_s))] ds,$$

where m is his own stationary distribution.

Interpretation of MFG (2)

The MFG system can be intended as an optimality condition for the following control of PDE problem:

The overall population controls his own stationary distribution via the Kolmogorov equation

$$-\Delta m + \mathrm{div}(\alpha\, m) = 0$$

and aims at minimizing

$$\overline{\mathcal{J}}(\alpha) = \int m L(\alpha) + F(x, m),$$

where $\partial_m F(x, m) = f(x, m)$.

2. Quadratic MFG and Hopf-Cole

Suppose now that

$$\boxed{H(p) = \frac{1}{2}|p|^2.}$$

(this is reasonable)

Then the MFG system reads

$$\begin{cases} \text{(HJB)} & -\Delta u + \frac{1}{2}|Du|^2 + \lambda = f(x, m), \quad \text{in } \Omega, \\ \text{(K)} & -\Delta m - \operatorname{div}(Du\, m) = 0, \\ & \int_\Omega m = 1, \end{cases}$$

Observe that $m = c\, e^{-u}$ trivializes (K), $c > 0$.

and, substituting into (HJB),

$$-\Delta u + \frac{1}{2}|Du|^2 + \lambda = f(x, c\, e^{-u}), \quad in\ \Omega.$$

Moreover, if we set $\varphi = \sqrt{c\, e^{-u}} = \sqrt{m}$, then (HJB) becomes

$$-2\Delta\varphi + (f(x, \varphi^2) - \lambda)\varphi = 0 \quad in\ \Omega,$$

which is a (nonlinear eigenvalue problem for a)
semilinear elliptic equation, with

$$\int \varphi^2 = 1.$$

Such a transformation, called Hopf-Cole, reduces the stationary MFG system to a single equation. (Proposed in [Lasry, Lions, '06])

This is exploited, for example, in

- [Gueant, '09], [Gueant, '12], quadratic models and numerical analysis,
- [Cardaliaguet, Lasry, Lions, Porretta '12], long-time average of MFG,
- [Gomes, Sanchez Morgado, '14], stochastic Evans-Aronsson problem.
- [Gomes, Pimentel, '14], logarithmic costs.

3. A generalization of Hopf-Cole

If the Hamiltonian is quadratic, a pointwise relation between u and m trivializes (K) and, $m^{1/2}$ transforms (HJB).

Now, what about the NON-QUADRATIC case?? i.e.

$$\boxed{H(p) = \frac{1}{r'}|p|^{r'}, \quad r' > 1}$$

which is the Leg. transform of $L(\alpha) = \frac{1}{r}|\alpha|^r$, $r' = \frac{r}{r-1}$.

Is there a way to rewrite MFG in a simpler form?

Theorem

Suppose that H is non-quadratic. Let (u, m, λ) be a (pointwise) solution of the MFG system. It the following equality holds,

$$Dm + m|Du|^{r'-2}Du = 0 \quad \text{in } \Omega, \tag{$\star$}$$

then

$$\varphi := m^{\frac{1}{r}}$$

is a (positive) solution of

$$\boxed{-r^{r-1}\Delta_r\varphi + (f(x, \varphi^r) - \lambda)\varphi^{r-1} = 0 \quad in\ \Omega.}$$

Here $\Delta_r\varphi = \mathrm{div}(|D\varphi|^{r-2}D\varphi)$.

So, IF the pointwise relation $(\star)$ between Du and m, Dm holds, (K) is trivialized, and $m^{1/r}$ transforms (HJB).

Remarks

- What is $(\star)$? Recall that (K) is

$$\mathrm{div}[Dm + m|Du|^{r'-2}Du] = 0 \quad \text{in } \Omega$$

- It works also for weak solutions of the MFG system.
- The transformation goes similarly the other way round.
- It is the standard Hopf-Cole if $r = 2$, $(\star)$ is always satisfied.

So, IF the pointwise relation $(\star)$ between Du and m, Dm holds, (K) is trivialized, and $m^{1/r}$ transforms (HJB).

Remarks

- What is $(\star)$? It comes from (K): the *div*-free vector field is zero.

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Proof

$(\star) \Rightarrow rD\varphi = -|Du|^{r'-2} Du\, \varphi.$
Now, by an easy substitution,

$$r^{r-1}\Delta_r \varphi = r^{r-1}\mathrm{div}(|D\varphi|^{r-2}D\varphi) = \mathrm{div}(|Du|^{(r'-1)(r-2)+r'-2}Du\,\varphi^{r-1})$$

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$$
\begin{aligned}
r^{r-1}\Delta_r\varphi = r^{r-1}\mathrm{div}(|\varphi|^{r-2}\varphi) &= \mathrm{div}(Du\,\varphi^{r-1}) \\
&= \varphi^{r-1}(\Delta u + (r-1)D\varphi\cdot Du) \\
&= \varphi^{r-1}(\Delta u - \frac{r-1}{r}|Du|^{r'}) \\
&\overset{\text{(HJB)}}{=} \varphi^{r-1}(\lambda - f(x,m)) \\
&= \varphi^{r-1}(\lambda - f(x,\varphi^r)).
\end{aligned}
$$

■

Proof (with a bit of insight)

$(\star) \Rightarrow D\varphi = -\frac{\varphi}{r} DH(Du)$.
Now, by an easy substitution,

$$
\begin{aligned}
r^{r-1}\Delta_r \varphi = r^{r-1}\mathrm{div}(DL(D\varphi)) &= r^{r-1}\mathrm{div}\left(DL\left(-\frac{\varphi}{r}DH(Du)\right)\right) \\
&= \mathrm{div}\left(\varphi^{r-1}DL\left(DH(Du)\right)\right) \\
&= \mathrm{div}\left(\varphi^{r-1}Du\right) \\
&= \ldots
\end{aligned}
$$

as DL is homogeneous and $DL = DH^{-1}$.

∎

(If L is not homogeneous, this trick is likely to fail).

A (formal) variational explanation

We observed that the a MFG system can be seen as the optimality condition for the control of PDE problem

$$\mathrm{div}[-Dm + \alpha\, m] = 0, \quad \int m = 1$$

with cost

$$\overline{\mathcal{J}}(\alpha) = \frac{1}{r} \int m|\alpha|^r + F(x, m).$$

If α^* is an optimal control, and $-Dm^* + \alpha^*\, m^* = 0$ (remember $(\star)$?), then

$$\alpha^* = \frac{Dm^*}{m^*}$$

and m^* is a minimizer of

$$\overline{\mathcal{J}}(m) = r^{r-1} \int |D(m^{1/r})|^r + F(x, m).$$

Hence, setting $\varphi := m^{\frac{1}{r}}$, φ is a minimizer of

$$\overline{\mathcal{J}}(\varphi) = r^{r-1} \int |D\varphi|^r + F(x, \varphi^r),$$

under the constraint $\int \varphi^r = 1$, which is then a solution of

$$-r^{r-1}\Delta_r \varphi + f(x, \varphi^r)\varphi^{r-1} = \lambda \varphi^{r-1} \quad \textit{in } \Omega.$$

We now move to a crucial question: is $(\star)$ ever satisfied (when $r \neq 2$)?

Yes. We need to specify some information, for example some symmetry or some suitable boundary conditions.

Corollary

Suppose that $\Omega = \{x \in \mathbb{R}^d : |x| < R\}$ for some $R \in (0, \infty]$.
Then, (u, m, λ) is a radial solution of the MFG system if and only if (φ, λ) is a radial solution of the Δ_r equation.

Corollary

Suppose that $\Omega = (a, b)$ for some $-\infty < a < b < \infty$.
Then, (u, m, λ) is a radial solution of the MFG system and $u'(a) = u'(b) = m'(a) = m'(b) = 0$ if and only if (φ, λ) is a radial solution of the Δ_r equation and $\varphi'(a) = \varphi'(b) = 0$.

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4. An application

Uniqueness of radial solutions of MFG systems in $\mathbb{R}^d$.

joint work with D. Castorina

We are interested in uniqueness results when the cost function is

$$\boxed{f(x, m) = -m^{\frac{\sigma}{r}}, \quad \sigma > 0, (r > 1),}$$

i.e. aggregation is promoted.

It is well-known that if the cost is increasing w.r.t. m, then uniqueness of solutions holds [Lasry,Lions 06].

Note that our f is decreasing w.r.t. m, here very little is known.

Moreover, f is unbounded, which might be an issue for existence.

In the periodic setting, $\Omega = [0,1]^d$:
uniqueness of solutions does not hold (see, again, [Lasry,Lions 06]): presence of a constant and a non-constant solution.

In the unbounded setting, $\Omega = \mathbb{R}^d$:
if $f(x, m) = -\log(m)$, it is shown in [Gueant, '09] that the explicit solution is unique up to translations.

our goal: exploit the connection

$$\text{MFG system} \longleftrightarrow \Delta_r \text{ equation}$$

to better understand the problem.

Theorem

Let $r \geq 2$. Then, there exist $0 < \hat{\sigma} < \sigma^*$ which depend on r, d such that if $\sigma \in (0, \sigma^*)$ and $\sigma \neq \hat{\sigma}$, radial solutions[a] of the MFG system are unique up to translations.

[a] By solution we mean $u \in C^2$, $0 < m \in C^1$ such that $m \to 0$ as $x \to \infty$

If $r = 2$, then uniqueness holds also for non-radial solutions.

Proof

By the preceding discussion, given two radial solutions (u_i, m_i, λ_i), $i = 1, 2$ of the MFG system, $\varphi_i := {m_i}^{1/r}$ solve

$$-r^{r-1}\Delta_r \varphi_i = \lambda_i {\varphi_i}^{r-1} + {\varphi_i}^{\sigma+r-1} \quad in\ \Omega,$$

and $\psi_i(x) := |\lambda_i|^{\frac{1}{2-\sigma-r}} \varphi_i(|\lambda_i|^{\frac{\sigma}{r(2-\sigma-r)}} x)$ solve

$$-r^{r-1}\Delta_r \psi_i = -{\psi_i}^{r-1} + {\psi_i}^{\sigma+r-1} \quad in\ \Omega,$$

Ground states of this equation are unique up to translations (see, for example, [Pucci, Serrin 98], [Serrin, Tang 2000]), in particular $\int \psi_1^r = \int \psi_2^r$ and $\int {\varphi_1}^r = \int {\varphi_2}^r = 1$ imply

$$|\lambda_1|^{r^2-d\sigma} = |\lambda_2|^{r^2-d\sigma},$$

which tells us that $\varphi_1(\cdot) = \varphi_2(\cdot + y)$ for some $y \in \mathbb{R}^d$ if $\sigma \neq \frac{r^2}{d} =: \hat{\sigma}$. ■

Actually, solutions of the MFG system are obtained by a suitable scaling of

$$-r^{r-1}\Delta_r \psi = -\psi^{r-1} + \psi^{\sigma+r-1}.$$

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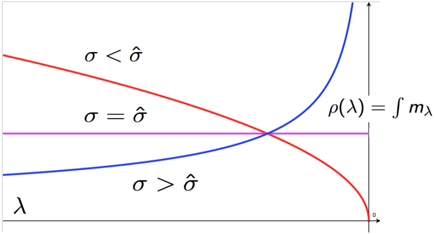
If we plot $\lambda \mapsto \rho(\lambda) = \int m_\lambda$



we realize that

- If $\sigma \neq \hat{\sigma}$, for all $\rho > 0$ there exists a unique radial solution (up to transl.) of MFG such that $\int m = \rho$.
- If $\sigma = \hat{\sigma}$ there exists a continuum of radial solutions of MFG such that $\int m = \hat{\rho}$, for some $\hat{\rho} > 0$.

Uniqueness with a potential?

Let $V(|x|) : \mathbb{R}^d \to \mathbb{R}$ be an increasing in $|x|$ and coercive radial potential. What happens if we set the cost function f to be

$$\boxed{f(x, m) = -m^{\frac{\sigma}{r}} + V(|x|),}$$

namely if we localize the problem?

We expect m to concentrate at the minimum point of V, ruling out the uniqueness of equilibria "up to translations".

As before, we move our attention to the transformed MFG system

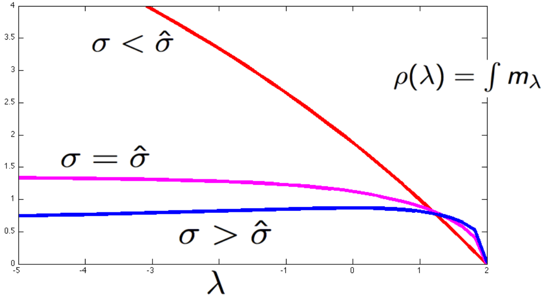
$$-r^{r-1}\Delta_r\varphi = (\lambda - V(|x|))\varphi^{r-1} + \varphi^{\sigma+r-1} \quad \textit{in } \Omega.$$

Due to the presence of V,

- the equation looses scaling invariance,
- the problem becomes compact,

What do we expect?

Numerically,



Conjecture

- If $\sigma < \hat{\sigma}$, for all $\rho > 0$ there exists a unique (radial) solution of MFG such that $\int m = \rho$.
- If $\sigma = \hat{\sigma}$, for all $0 < \rho < \rho_1$ there exists a unique (radial) solution of MFG such that $\int m = \rho$.
- If $\sigma > \hat{\sigma}$,
  - for all $0 < \rho \leq \rho_2$ there exist at least two (radial) solutions of MFG such that $\int m = \rho$.
  - for all $\rho > \rho_2$ there are no (radial) solutions of MFG such that $\int m = \rho$.

Conclusions

- Hopf-Cole transformations are useful tools for (preliminary) analysis of MFG systems.
- What if $(\star)$ does not hold? Maybe φ satisfies a more general equation when $r \neq 2$.
- For non-stationary (and non-quadratic) MFG systems, the presence of time derivatives seems to be against an effective general Hopf-Cole.

Thanks for your attention !