Statistical Inference based on Inverse Data Generating Equation
(Generalized Fiducial Inference)

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**Oxford English Dictionary**

- **adjective TECHNICAL** (of a point or line) used as a fixed basis of comparison.
- **Origin** from Latin fiducia ‘trust, confidence’

**Merriam-Webster dictionary**

1. taken as standard of reference *a fiducial mark*
2. founded on faith or trust
3. having the nature of a trust : FIDUCIARY
Fisher (1930) introduced the idea of fiducial probability and inference in an attempt to overcome what he saw as a serious deficiency of the Bayesian approach to inference – use of a prior distribution when no prior information was available.

$$r(\xi|x) = -\frac{\partial F(x|\xi)}{\partial \xi}.$$
Brief history of fiducial inference

Fisher (1930) introduced the idea of fiducial probability and inference in an attempt to overcome what he saw as a serious deficiency of the Bayesian approach to inference – use of a prior distribution when no prior information was available.

\[ r(\xi|x) = -\frac{\partial F(x|\xi)}{\partial \xi} . \]

Fisher (1935) further elaborated on this idea. E.g., to eliminate nuisance parameters he suggested substituting their fiducial distribution. As an example he considered the inference for the difference of two normal means – “Behrens-Fisher problem”.
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- Objective Bayesian inference; choice of $\pi(\theta)$ when we have no prior info, e.g., reference prior **Berger, Bernardo & Sun (2009)**.
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- **Inferential Models;**[Liu, Martin](https://www.jstor.org/stable/2273812) and coworkers.

- **Confidence Distributions;**[Xie, Singh & Strawderman (2011)](https://doi.org/10.1002/jse.2497), [Schweder & Hjort (2002)](https://doi.org/10.1093/biomet/89.1.35). The idea is to use a frequentist procedure (e.g., one sided CI for all possible confidence levels $\alpha$) to define a distribution on the parameter space.

- **Objective Bayesian inference;** choice of $\pi(\theta)$ when we have no prior info, e.g., reference prior [Berger, Bernardo & Sun (2009)](https://doi.org/10.1002/9780470316179). With improper reference prior one needs to prove that the posterior is a proper distribution on an individual basis.
Bird’s Eye of Statistical Inference

We are given a data set $X$ and are asked to provide some information about the mechanism used to generate it.
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Assume that the data was generated using a model $\mathcal{P} = \{P_\theta\}_{\theta \in \Theta}$.

Goal is to find a $P_\theta$'s that best fit the data with possible some additional considerations, e.g., sparsity.
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- **Frequentist Inference**
  - Assume that the data was generated using a model $\mathcal{P} = \{P_{\theta}\}_{\theta \in \Theta}$.
  - Goal is to find a $P_{\theta}$'s that best fit the data with possible some additional considerations, e.g., sparsity.
  - Each statistical problem requires its own solution and the quality of the solution is judged by repeated sampling performance **(Cournot's principle)**.
Bayesian inference

It is assumed that the value \( \theta \in \Theta \) was generated using some known distribution \( \pi(\theta) \), prior, and we have only single, fully known distribution \( P_\theta \cdot \pi(\theta) \).
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There is only one solution for each statistical problem. The remaining problem specific issue is to find the solution computationally and to select the right model + prior.
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  - Philosophical interpretation of fiducial probability is obscure.
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We use fiducial distribution to propose statistical methods (e.g., confidence Intervals) and then evaluate the methods using repeated sampling performance.
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  - Goal is to find a distribution on the parameter space \( \Theta \) that in summarizes the information we have obtained from the data.
  - Philosophical interpretation of fiducial probability is obscure.
  - We use fiducial distribution to propose statistical methods (e.g., confidence Intervals) and then evaluate the methods using repeated sampling performance.
  - The fiducial distribution is usually not a posterior with respect to any (data independent) prior (Grundy, 1956).
The aim of this talk

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- We attempt to strip down all layers of additional structure.
- Our definition does not produce a "unique fiducial distribution". Regardless, the fiducial distribution is always proper.
The aim of this talk

- We explain the definition of fiducial distribution as we generalize it demonstrating it on several examples.
  - Applicable to both discrete and continuous distributions.
  - We attempt to strip down all layers of additional structure.
  - Our definition does not produce a "unique fiducial distribution". Regardless, the fiducial distribution is always proper.

- We proved some asymptotic theorems justifying this method of deriving inference procedures. Simulations usually show very good frequentist performance.
Comparison to MLE

- **Density** is the function $f(x, \xi)$, where $\xi$ is fixed and $x$ is variable.
Comparison to MLE

- **Density** is the function $f(\mathbf{x}, \xi)$, where $\xi$ is fixed and $\mathbf{x}$ is variable.
- **Likelihood** is the function $f(\mathbf{x}, \xi)$, where $\xi$ is variable and $\mathbf{x}$ is fixed.
Comparison to MLE

Consider the data generating (structural) equation

\[ X = T(U, \xi), \]

- \( U \) is a random variable/vector with known distribution
- \( \xi \) is a fixed parameter.
- The distribution of the data \( X \) is implied from \( U \) via the structural equation. I.e., one can generate \( X \) by generating \( U \) and plugging it into the structural equation.
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After observing \( X = x \) deduce a distribution for \( \xi \) from that of \( U \) via the structural equation. I.e., generate \( \xi \) by generating \( U^* \), plugging it into the structural equation and solving for \( \xi \).
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If the solution does not exist, discard this value of \( U^* \), i.e., condition the distribution of \( U \) on the fact that the solution exists.
Example – binomial

Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli$(p)$. Therefore

$$X_i = I_{[0,p)}(U_i), \quad i = 1, \ldots, n,$$

where $U_i$ are i.i.d. uniform.
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Define the inverse image of $T$

$$Q(x_1, \ldots, x_n, u_1, \ldots, u_n) = \{ p : x_i = I_{[0,p)}(u_i) \} = (m, M),$$

where

$$m = \max_{x_i = 1} u_i; \quad M = \min_{x_i = 0} u_i.$$
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The fiducial distribution is

$$Q(x_1, \ldots, x_n, U_1^*, \ldots, U_n^*) \mid Q(x_1, \ldots, x_n, U_1^*, \ldots, U_n^*) \neq \emptyset \quad \mathcal{D} \equiv (U^{* \sum x_i : n}; U^{* (1+\sum x_i) : n})$$
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$$\overset{D}{=} (U_{(\sum x_i):n}, U_{(1+\sum x_i):n})$$

We need to select a point inside the interval. We recommend selecting each edge with equal probability.
Single Normal

- Consider \( X = \mu + Z \), where \( Z \sim N(0, 1) \).
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Observe \( X = 10 \). Then we have \( 10 = \mu + Z \).
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**Fiducial argument:**

$$P(\mu = 3 \pm dx) = P(10 - Z = 3 \pm dx) = P(Z = 7 \pm dx) \approx 1.83 \cdot 10^{-11} \, dx$$
Consider $X = \mu + Z$, where $Z \sim N(0,1)$.

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We can simulate this distribution using $R_\mu = 10 - Z^*$, where $Z^* \sim N(0, 1)$ independent of $Z$. 

Location Normal

Consider $X_i = \mu + Z_i$ where $Z_i$ are i.i.d. $N(0, 1)$. 
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Observe $(x_1, \ldots, x_n)$. We cannot simply follow the previous idea of setting $\mu = x_1 - Z_1^*, \ldots, \mu = x_n - Z_n^*$. 
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The fiducial distribution can be defined as

$$x_1 - Z_1^* | x_2 = x_1 - Z_1^* + Z_2^*, \ldots, x_n = x_1 - Z_1^* + Z_n^*.$$
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After simplification the fiducial distribution is \( N(\bar{x}, 1/n) \).

We have non-uniqueness due to Borel paradox.
Remarks

There are three challenges in the definition of generalized fiducial distribution.
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  - The choice among multiple solutions:

    - Arises if the inverse image \( Q(x, U^*) \) has more than one element but disappears asymptotically for parametric problems.
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- The choice among multiple solutions:
  - Arises if the inverse image \( Q(x, U^*) \) has more then one element but disappears asymptotically for parametric problems.

- The conditioning on the fact that solution exist:
  - Arises if \( P\{Q(x, U^*) \neq \emptyset\} = 0 \) – Borel paradox.
  - “Resolved by fat data”.
Fat data

Borel paradox was caused by the fact that probability of observing our data could be 0.
Fat data

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- Due to instrument limitations we never observe our data exactly.
**Fat data**

- Borel paradox was caused by the fact that probability of observing our data could be 0.
- Due to instrument limitations we never observe our data exactly.
  - My height is $1.85 < x_i < 1.86$.
- Any number stored on a computer is known only to a machine precision.
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- Due to instrument limitations we never observe our data exactly.
  - My height is $1.85 < x_i < 1.86$.
  - Any number stored on a computer is known only to a machine precision.
- We derive generalized fiducial distribution directly for discretized data or take a limit as the discretization refines.
Fat data on a diet

Assume that the data vector $x \in \mathbb{R}^n$ has continuous distribution with the parameter $\xi \in \mathbb{R}^p$
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Assume that the data vector $\mathbf{x} \in \mathbb{R}^n$ has continuous distribution with the parameter $\mathbf{\xi} \in \mathbb{R}^p$

Interpret fiducial recipe as the weak limit (as $\varepsilon \downarrow 0$) of

$$\arg\min_{\mathbf{\xi}} \| \mathbf{x} - T(\mathbf{U}^*, \mathbf{\xi}) \| \mid \{ \min_{\mathbf{\xi}} \| \mathbf{x} - T(\mathbf{U}^*, \mathbf{\xi}) \| < \varepsilon \}$$  (1)
Assume that the data vector $x \in \mathbb{R}^n$ has continuous distribution with the parameter $\xi \in \mathbb{R}^p$.

Interpret fiducial recipe as the weak limit (as $\varepsilon \downarrow 0$) of

$$\arg \min_{\xi} \|x - T(U^*, \xi)\| \mid \{\min_{\xi} \|x - T(U^*, \xi)\| < \varepsilon\}$$

The condition in (1) uses data fattened to a ball

$$\mathcal{B}_\varepsilon(x) = \{y : \|x - y\| < \varepsilon\}$$
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$$

- The condition in (1) uses data fattened to a ball

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\mathcal{B}_\varepsilon(x) = \{y : \|x - y\| < \varepsilon\}
$$

- Similar to the idea of ABC; generating from prior replaced by $\min$. 

Theoretical results

If using $\| \|_\infty$ and smooth $T$ the limiting conditional distribution (1) has density (Hannig, 2012)

$$r(\xi|x) = \frac{f_x(x|\xi)J(x, \xi)}{\int_{\Xi} f_x(x|\xi')J(x, \xi') d\xi'},$$
Theoretical results

If using $\| \|_\infty$ and smooth $T$ the limiting conditional distribution (1) has density (Hannig, 2012)

$$r(\xi|\mathbf{x}) = \frac{f_X(\mathbf{x}|\xi) J(\mathbf{x}, \xi)}{\int_{\Xi} f_X(\mathbf{x}|\xi') J(\mathbf{x}, \xi') \, d\xi'},$$

where $J(\mathbf{x}, \xi) = \sum_{i=(i_1, \ldots, i_p)} \left| \det \left( \frac{d}{d\xi} T(u, \xi) \bigg|_{u=T^{-1}(\mathbf{x}, \xi)} \right)_{i_1} \right|$ and $(A)_i$ is the $p \times p$ matrix comprising of the $i_1, \ldots, i_p$th row of the $n \times p$ matrix $A$. 
Let $X_i = F^{-1}(\xi, U_i)$ be cont. with density $f(x|\xi)$.

Then $J(x, \xi) = \sum_{i=(i_1,\ldots,i_p)} \frac{|\det(\frac{d}{d\xi} F(x_i,\xi))|}{\prod_i f(x_i,\xi)}$.
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Then $J(x, \xi) = \sum_{i=(i_1, \ldots, i_p)} \left| \frac{\text{det} \left( \frac{d}{d\xi} F(x_i; \xi) \right)}{\prod f(x_i; \xi)} \right|$.

Often $\binom{n}{p}^{-1} J(x, \xi) \to E_{\xi_0} \left[ \frac{\text{det} \left( \frac{d}{d\xi} F(X_i; \xi) \right)}{\prod f(x_i; \xi)} \right]$ providing an empirical Bayes interpretation.
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Confidence intervals based on generalized fiducial distribution are often correct asymptotically because of “Bernstein-von Mises” theorem for fiducial distributions Hannig (2009, 2012), Sonderegger & Hannig (2012).
Theoretical result for discretized data

Assume structural equation \( X_i = F^{-1}(U_i, \xi) \)

- \( \xi \) is \( p \) dimensional and \( U_i \) are i.i.d. \( U(0, 1) \).
- \( F(x, \xi) \) is continuously differentiable in \( \xi \) for all \( x \).
- \( (F(x_1, \xi), \ldots, F(x_p, \xi)) = (u_1, \ldots, u_p) \), taken as a function of \( \xi \) is one to one for each \( x \).
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- $\xi$ is $p$ dimensional and $U_i$ are i.i.d. $U(0, 1)$.
- $F(x, \xi)$ is continuously differentiable in $\xi$ for all $x$
- $(F(x_1, \xi), \ldots, F(x_p, \xi)) = (u_1, \ldots, u_p)$, taken as a function of $\xi$ is one to one for each $x$.

Data were discretized to a fixed partition $(-\infty, a_1], (a_1, a_2], \ldots, (a_k, \infty)$.

- $P(X \in (a_j, a_{j+1}]) > 0$ for all $j$.
- For all $j \subset \{1, \ldots, k\}$, the Jacobian $\det \left( \frac{dF(a_j, \xi_0)}{d\xi} \right) \neq 0$. 

Theoretical result for discretized data

Assume structural equation $X_i = F^{-1}(U_i, \xi)$

- $\xi$ is $p$ dimensional and $U_i$ are i.i.d. $U(0, 1)$.
- $F(x, \xi)$ is continuously differentiable in $\xi$ for all $x$.
- $(F(x_1, \xi), \ldots, F(x_p, \xi)) = (u_1, \ldots, u_p)$, taken as a function of $\xi$ is one to one for each $x$.

Data were discretized to a fixed partition

$(-\infty, a_1], (a_1, a_2], \ldots, (a_k, \infty)$.

- $P(X \in (a_j, a_{j+1}]) > 0$ for all $j$.
- For all $j \subset \{1, \ldots, k\}$, the Jacobian $\det \left( \frac{dF(a_j; \xi)}{d\xi} \right) \neq 0$.

**Theorem** (Hannig (2012)). *Confidence sets based on the generalized fiducial distribution will have asymptotically correct coverage as number of data points goes to infinity and resolution remains fixed.*
Model Selection

Consider several models \((\mathcal{M}_i)_{i \in I}\), each with data generating equation \(X = T_i(U, \xi), i \in I\).
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If all models have the same dimension (number of parameters), the marginal distribution of $i$th model is

$$r(M_i) \propto \int_{\Xi} f_i(x|\xi) J_i(x, \xi) d\xi,$$

where $f_i$ and $J_i$ were computed using the $i$th model.
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When the number of parameters is different, penalty is needed.

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r(\mathcal{M}_i) \propto e^{-q(i)} \int_{\Xi} f_i(x|\xi) J_i(x, \xi) \, d\xi;
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we choose the Minimum Description Length penalty for \(q(i)\).
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Key comparison

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- For simplicity assume, each laboratory reports a confidence interval based on a $T$ distribution, measuring the same object.

- Goal is to combine the intervals in a way that down ways potential outliers. Outright dropping of odd results is politically not feasible.
Key comparison - our solution

- All possible subsets of laboratories are considered as measuring the correct values with the rest considered as outliers.
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- All possible subsets of laboratories are considered as measuring the correct values with the rest considered as outliers.
- For each model the fiducial distributions measuring the true value are combined using Hannig & Xie (2012) and the fiducial model probability is computed.

\[
    r(\mu) \propto \sum_{j \in I} C(\mathcal{M}_j) \sum_{i \in \mathcal{M}_j} \left\{ \frac{1}{n_i} + \frac{(\mu - \bar{x}_i)^2}{(n_i - 1)s_i^2} \right\}^{-1/2} \prod_{i \in \mathcal{M}_j} \left\{ 1 + \frac{n_i(\mu - \bar{x}_i)^2}{(n_i - 1)s_i^2} \right\}^{-(n_i-1)/2} \\
    \times e^{-(k - |\mathcal{M}_i|) \log(SSE)/2}
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Preliminary simulation using importance sampling shows somewhat conservative performance.
Key comparison - example
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Example - Linear Mixed Model

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There seems to be no (non-Bayesian) unified approach producing good quality confidence sets. Most procedures in the literature are designed to solve special cases Burdick, Graybill, Wang or use insufficient statistics Khuri, Mathews and Sinha.

We will propose a procedure that produces confidence sets for large class of linear mixed models. Additionally it allows for discretized data.
Linear Mixed Model

Consider a structural equation

\[ Y = X\beta + \sum_{i=1}^{k} \sigma_i \sum_{j=1}^{l_k} V_{i,j} Z_{i,j} \]
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- $Y$ observations, $X$ design matrix, $\beta$ fixed effect parameters
- $k$ number of random effects, $l_i$ number of levels per effect,
- $V_{i,j}$ var component design vectors, $\sigma_i^2$ variance of the $i$th effect
- $Z_{i,j}$ i.i.d. $N(0, 1)$
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Contains a wide variety of linear mixed models.
Linear Mixed Model

\[ \mathbf{Y} = \mathbf{X}\beta + \sum_{i=1}^{k} \sigma_i \sum_{j=1}^{l_k} V_{i,j} \mathbf{Z}_{i,j} \]

- Linear regression
  - \( k = 1, l_1 = n, V_{1,:} = (V_{1,1}, \ldots, V_{1,n}) = I \)
  - \( m \) regression coefficients, \( \sigma_1^2 \) error variance
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- **One way random effects model**
  - \( X = 1, k = 2, l_1 \) number of levels for random effect, \( l_2 = n \), \( V_{1,i} \) indicates which observations are in group \( i \), \( V_{2,.} = I \)
  - \( m \) overall mean, \( \sigma_1^2 \) random effect variance, \( \sigma_2^2 \) error variance
Linear Mixed Model

Assume we observe $a \leq Y \leq b$
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- We can generate \( Z_{i,j}^{*} \) as i.i.d. \( N(0,1) \) and solve for \( \beta, \sigma \)
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Possibilities include

- Gibbs sampler – does not mix well if there is too much precision.
- Simulated tampering – works but slow
- We proposed a particular implementation of Sequential Monte Carlo algorithm – works well if the number of parameters is reasonable (\(< 10\)).
Simulation study

One-way random effects: $Y_{ijk} = \mu + \alpha_i + \epsilon_{ij}$

($\mu$ fixed, $\alpha$ and $\epsilon$ are independent and $\sim$ Normal)
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- We considered a number of models with various levels of imbalance and values of parameters.
95% CI for random effects (nested)
95% CI for random effects (crossed)
Some generalized fiducial projects

We applied generalized fiducial inference to:
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  + Linear Mixed Model
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- Confidence sets for wavelet regression using fiducial idea. This is related to model selection.
- Extreme value data (Generalized Pareto) & Maximum mean (QT intervals) and model comparison.
- Ultra-highdimensional Regression Model (How to properly introduce a penalty?)
Concluding remarks

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- Many simulation studies show that generalized fiducial solutions have very good small sample properties.

- Current popularity of generalized inference in some applied circles suggests that if computers were available 70 years ago, fiducial inference might not have been rejected.
Quotes

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