Mean field games with congestion

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Outline

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1. Introduction
2. Stationary problems
   - Variational mean-field games
   - Congestion models
3. Time dependent problems
   - Logarithmic nonlinearity
   - Short-time existence for congestion problems

Collaborators

- H. Mitake (Hiroshima)
- H. S. Morgado (UNAM, Mexico)
- S. Patrizi (Berlin)
- E. Pimentel
- V. Voskanyan (KAUST)

Standard MFG in reduced form

- Time dependent MFG

$$\begin{cases} -u_t + H(Du, x) = \Delta u + F(m) \\ m_t - \operatorname{div}(D_p H m) = \Delta m \end{cases}$$

with $m(x, 0)$ and $u(x, T)$ given.

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Typical non-linearity F:

- Non-local: $F(m) = G(\eta * m)$.
- Power-like: $F(m) = m^{\alpha}$.
- Logarithm: $F(m) = \ln m$.

Typical Hamiltonian: $H(x, p) = a(x)(1 + |p|^2)^{\gamma/2} + V(x)$, a, V periodic, smooth, $a > 0$.

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A more general case

Then the value function u solves the Hamilton-Jacobi equation

$$-u_t + H(D_x u, x, m) = \frac{\sigma^2}{2}\Delta u$$

and m solves

$$m_t - \text{div}(D_p H m) = \frac{\sigma^2}{2}\Delta m.$$

Boundary data

The value function u and the probability measure m satisfy initial-terminal problem

$$u(x,T)=\psi(x)\qquad m(x,0)=m_0(x).$$

To simplify the presentation we work in the spatially periodic setting. That is $x\in\mathbb{T}^d$, the standard d-dimensional torus identified with $[0,1]^d$.

The solution u to the Hamilton-Jacobi equation is a value function:

$$u(x,t) = \inf_{\mathbf{v}} E \int_t^T L(\mathbf{x}, \mathbf{v}, m(\cdot, s)) ds + \psi(\mathbf{x}(T), m(\cdot, T)),$$

where the infimum is taken, over all progressively measurable controls $\mathbf{v}$ (w.r.t. the Brownian filtration of W_t), and

$$d\mathbf{x} = \mathbf{v} dt + \sigma dW_t.$$

Congestion

At the level of the Lagrangian, two important congestion models are:

- Logarithmic non-linearity

$$L(x, v, m) = L_0(x, v) + \ln m,$$

which makes low density regions extremely desirable.

- Standard congestion problem:

$$L(x, v, m) = m^{\alpha} L_0(x, v),$$

which makes very expensive to move in high-density areas.

These two Lagrangians give yield the systems

- Logarithmic non-linearity

$$\begin{cases} -u_t + H(x, Du) = \ln m + \Delta u \\ m_t - div(D_pHm) = \Delta m. \end{cases}$$

- Standard congestion

$$\begin{cases} -u_t + \frac{|Du|^2}{m^\alpha} = \Delta u \\ m_t - div(m^{1-\alpha}Du) = \Delta m. \end{cases}$$

Research goals

- Uniqueness (solved under general conditions by Lions)
- **Existence**
- **Regularity**

Outline

Consider the variational problem

$$\min \int_{\mathbb{T}^d} e^{H(Du,x)} dx.$$

The Euler-Lagrange equation is the Mean Field Game:

$$\begin{cases} H(Du,x) = \ln m + \overline{H} \\ -\operatorname{div}(D_p H m) = 0, \end{cases}$$

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As proved by L. C. Evans - first order MFG with quadratic-growth Hamiltonians admit classical solutions.

The stochastic Evans-Aronsson problem is the problem

$$\min_u \int_{\mathbb{T}^d} e^{-\Delta u + H(Du,x)}.$$

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- In low dimension $d \leq 3$ under quadratic growth conditions we (G. and Sanchez-Morgado) also obtained existence of a smooth solution.
- This result was subsequently improved by G. , Patrizi, Voskanyan for arbitrary dimension.

Running costs such as

$$L(x,v,m) = m^{\alpha}(x)\frac{|v|^2}{2} - V(x)$$

correspond to the congestion MFG:

$$\begin{cases} u + V(x) + \frac{|Du|^2}{2m^{\alpha}} = \Delta u \\ m - \operatorname{div}(m^{1-\alpha}Du) = \Delta m + 1 \end{cases}$$

Theorem (D. G., H. Mitake)

Under the previous hypothesis, there exists a classical solution (u, m) *with m bounded by below if* $0 < \alpha < 1$.

- The proof relies on an a-priori bound for $\frac{1}{m}$ in L^∞.
- This bound depends on an explicit cancellation.
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The a-priori bound

The key a-priori bound follows from the identity:

$$\int_{\mathbb{T}^d} \left[u - \Delta u + \frac{|Du|^2}{2m^\alpha} - V \right] \cdot \frac{1}{rm^r} \, dx$$
$$- \int_{\mathbb{T}^d} \left[m - \Delta m - \operatorname{div}\left(m^{1-\alpha} Du \right) \right] \cdot \frac{1}{(r+1-\alpha) m^{r+1-\alpha}} \, dx$$
$$= - \int_{\mathbb{T}^d} \frac{1}{(r+1-\alpha) m^{r+1-\alpha}} \, dx.$$

Which after some surprising cancellations yields

$$\int_{\mathbb{T}^d} \frac{1}{(r+1-\alpha)m^{r+1-\alpha}}\,dx + \int_{\mathbb{T}^d} \frac{|Du|^2}{4rm^{r+\alpha}}\,dx + \int_{\mathbb{T}^d} \frac{|Dm|^2}{m^{r+2-\alpha}}\,dx$$
$$\leq \int_{\mathbb{T}^d} \frac{C}{rm^r}\,dx + \int_{\mathbb{T}^d} \frac{C}{(r-\alpha)m^{r-\alpha}}\,dx.$$

Outline

Initial-terminal value problem

$$-u_t + H(D_x u, x) = \frac{\sigma^2}{2}\Delta u + \ln m$$

$$m_t - \operatorname{div}(D_p H m) = \frac{\sigma^2}{2}\Delta m.$$

Together with smooth initial conditions for $m > 0$ and terminal conditions for u.

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Theorem (D. G., E. Pimentel)

The initial-terminal value problem for the logarithmic nonlinearity with Hamiltonians

$$H(x,p) = (1+|p|^2)^{\frac{\gamma}{2}} + V(x),$$

with $1 < \gamma < \frac{5}{4}$, *admits smooth solutions.*

The proof in three lines

- $\|u\| \leq C + C\|\ln m\|$
- $\|\ln m\| \leq C + C\|u\|^{\beta}$
- "result" ($\|u\|$ bounded) follows if $\beta < 1$.

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Key estimate

The key estimate is the following

$$\frac{d}{dt}\int_{\mathbb{T}^d}\frac{1}{m} \leq C\||D_pH|^2\|_{L^\infty}\int_{\mathbb{T}^d}\frac{1}{m}.$$

This estimate makes it possible to prove a bound

$$\int_{\mathbb{T}^d}|\ln m|^p \leq C + C\||D_pH|^2\|_{L^\infty}^p.$$

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The short time problem

- $H \sim |p|^\gamma$ for large p, and

$$\begin{cases} -u_t + m^\alpha H(x, \frac{Du}{m^\alpha}) = \Delta u \\ m_t - \operatorname{div}(D_p H(x, \frac{Du}{m^\alpha})m) = \Delta m. \end{cases}$$

- $u(x, T)$ and $m(x, 0)$ given
- T small.

Some a-priori bounds

Theorem (D. G. - V. Voskanyan)

Suppose $\alpha < \frac{2}{d-2}$ *and* $\gamma < 2$. *If* T *is small enough then* $\frac{1}{m} \in L^r$, *for any* r.

The proof of the theorem relies on the following estimate: Assume $\alpha < \frac{2}{d-2}$, $r > 1$, then there exists $q > r$ such that

$$\int_{\mathbb{T}^d} \frac{1}{m^r(x,t)} dx + \int_0^t \int_{\mathbb{T}^d} \left| D \frac{1}{m^{r/2}} \right|^2 dxdt + \int_0^t \int_{\mathbb{T}^d} \frac{|Du|^\gamma}{m^{r+\bar{\alpha}}} dxdt$$
$$\leq C + C \int_0^t \int_{\mathbb{T}^d} \frac{1}{m^q} dxdt.$$

Open questions

- General stationary congestion models
- Time dependent logarithmic non-linearity $\gamma \geq \frac{5}{4}$.
- Long time congestion problem.