Mean field games with congestion

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Outline

1. Introduction

2. Stationary problems
   - Variational mean-field games
   - Congestion models

3. Time dependent problems
   - Logarithmic nonlinearity
   - Short-time existence for congestion problems
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1 Introduction

2 Stationary problems
   - Variational mean-field games
   - Congestion models

3 Time dependent problems
   - Logarithmic nonlinearity
   - Short-time existence for congestion problems
Collaborators

- H. Mitake (Hiroshima)
- H. S. Morgado (UNAM, Mexico)
- S. Patrizi (Berlin)
- E. Pimentel
- V. Voskanyan (KAUST)
Standard MFG in reduced form

- **Time dependent MFG**
  \[
  \begin{aligned}
  -u_t + H(Du, x) &= \Delta u + F(m) \\
  m_t - \text{div}(D\rho Hm) &= \Delta m
  \end{aligned}
  \]
  with \(m(x, 0)\) and \(u(x, T)\) given.

- **Stationary version**
  \[
  \begin{aligned}
  H(Du, x) &= \Delta u + F(m) + H \\
  -\text{div}(D\rho Hm) &= \Delta m
  \end{aligned}
  \]
Standard MFG in reduced form

- **Time dependent MFG**
  \[
  \begin{cases}
  -u_t + H(Du, x) = \Delta u + F(m) \\
  m_t - \text{div}(D_pHm) = \Delta m
  \end{cases}
  \]

  with \( m(x, 0) \) and \( u(x, T) \) given.

- **Stationary version**
  \[
  \begin{cases}
  H(Du, x) = \Delta u + F(m) + \bar{H} \\
  - \text{div}(D_pHm) = \Delta m
  \end{cases}
  \]
Typical non-linearity $F$:

- Non-local: $F(m) = G(\eta \ast m)$.
- Power-like: $F(m) = m^{\alpha}$.
- Logarithm: $F(m) = \ln m$.

Typical Hamiltonian: $H(x, \rho) = a(x)(1 + |\rho|^2)^{\gamma/2} + V(x)$, $a$, $V$ periodic, smooth, $a > 0$. 
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A more general case

Then the value function $u$ solves the Hamilton-Jacobi equation

$$-u_t + H(D_x u, x, m) = \frac{\sigma^2}{2} \Delta u$$

and $m$ solves

$$m_t - \text{div}(D_p H m) = \frac{\sigma^2}{2} \Delta m.$$
The value function $u$ and the probability measure $m$ satisfy initial-terminal problem

$$u(x, T) = \psi(x) \quad m(x, 0) = m_0(x).$$

To simplify the presentation we work in the spatially periodic setting. That is $x \in \mathbb{T}^d$, the standard $d$-dimensional torus identified with $[0, 1]^d$. 
The solution $u$ to the Hamilton-Jacobi equation is a value function:

$$u(x, t) = \inf_{v} E \int_{t}^{T} L(x, v, m(\cdot, s)) ds + \psi(x(T), m(\cdot, T)),$$

where the infimum is taken, over all progressively measurable controls $v$ (w.r.t. the Brownian filtration of $W_t$), and

$$dx = vd t + \sigma dW_t.$$
At the level of the Lagrangian, two important congestion models are:

- Logarithmic non-linearity

\[ L(x, v, m) = L_0(x, v) + \ln m, \]

which makes low density regions extremely desirable.

- Standard congestion problem:

\[ L(x, v, m) = m^\alpha L_0(x, v), \]

which makes very expensive to move in high-density areas.
These two Lagrangians give yield the systems

- **Logarithmic non-linearity**

\[
\begin{align*}
-u_t + H(x, Du) &= \ln m + \Delta u \\
m_t - \text{div}(D_p H m) &= \Delta m.
\end{align*}
\]

- **Standard congestion**

\[
\begin{align*}
-u_t + \frac{|Du|^2}{m^\alpha} &= \Delta u \\
m_t - \text{div}(m^{1-\alpha} Du) &= \Delta m.
\end{align*}
\]
Research goals

- Uniqueness (solved under general conditions by Lions)
- Existence
- Regularity
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Consider the variational problem

$$\min \int_{T^d} e^{H(Du, x)} \, dx.$$ 

The Euler-Lagrange equation is the Mean Field Game:

$$\begin{cases} 
H(Du, x) = \ln m + \overline{H} \\
- \text{div}(D_p Hm) = 0,
\end{cases}$$

where the constant $\overline{H}$ is chosen so that $\int m = 1$. 

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where the constant $\bar{H}$ is chosen so that $\int m = 1$. 
As proved by L. C. Evans - first order MFG with quadratic-growth Hamiltonians admit classical solutions.
The stochastic Evans-Aronsson problem is the problem

$$\min_u \int_{\mathbb{T}^d} e^{-\Delta u + H(Du, x)}.$$

The Euler-Lagrange equation is

$$\begin{cases}
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The stochastic Evans-Aronsson problem is the problem

$$\min_u \int_{\mathbb{T}^d} e^{-\Delta u + H(Du, x)}.$$

The Euler-Lagrange equation is

\[\begin{cases}
-\Delta u + H(Du, x) = \ln m \\
-\Delta m - \text{div}(D\rho Hm) = 0.
\end{cases}\]
In low dimension $d \leq 3$ under quadratic growth conditions we (G. and Sanchez-Morgado) also obtained existence of a smooth solution.

This result was subsequently improved by G., Patrizi, Voskanyan for arbitrary dimension.
Running costs such as

$$L(x, v, m) = m^\alpha(x) \frac{|v|^2}{2} - V(x)$$

correspond to the congestion MFG:

$$\begin{cases} u + V(x) + \frac{|Du|^2}{2m^\alpha} = \Delta u \\ m - \text{div}(m^{1-\alpha}Du) = \Delta m + 1 \end{cases}$$
Theorem (D. G., H. Mitake)

Under the previous hypothesis, there exists a classical solution $(u, m)$ with $m$ bounded by below if $0 < \alpha < 1$. 
The proof relies on an a-priori bound for $\frac{1}{m}$ in $L^\infty$. This bound depends on an explicit cancellation. It is not known if similar results hold for general models or time-dependent problems.
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This bound depends on an explicit cancellation.
It is not known if similar results hold for general models or time-dependent problems.
The key a-priori bound follows from the identity:

\[
\begin{align*}
\int_{\mathbb{T}^d} \left[ u - \Delta u + \frac{|Du|^2}{2m^\alpha} - V \right] \cdot \frac{1}{rm^r} \, dx \\
- \int_{\mathbb{T}^d} \left[ m - \Delta m - \text{div} \left( m^{1-\alpha} Du \right) \right] \cdot \frac{1}{(r + 1 - \alpha)m^{r+1-\alpha}} \, dx \\
= - \int_{\mathbb{T}^d} \frac{1}{(r + 1 - \alpha)m^{r+1-\alpha}} \, dx.
\end{align*}
\]
Which after some surprising cancellations yields

\[
\int_{\mathbb{T}^d} \frac{1}{(r + 1 - \alpha)m^{r+1-\alpha}} \, dx + \int_{\mathbb{T}^d} \frac{|Du|^2}{4rm^{r+\alpha}} \, dx + \int_{\mathbb{T}^d} \frac{|Dm|^2}{m^{r+2-\alpha}} \, dx \\
\leq \int_{\mathbb{T}^d} \frac{C}{rm^r} \, dx + \int_{\mathbb{T}^d} \frac{C}{(r - \alpha)m^{r-\alpha}} \, dx.
\]
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Initial-terminal value problem

\[-u_t + H(D_x u, x) = \frac{\sigma^2}{2} \Delta u + \ln m\]

\[m_t - \text{div}(D_p Hm) = \frac{\sigma^2}{2} \Delta m.\]

Together with smooth initial conditions for \(m > 0\) and terminal conditions for \(u\).
Initial-terminal value problem

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Together with smooth initial conditions for \(m > 0\) and terminal conditions for \(u\).
Theorem (D. G., E. Pimentel)

The initial-terminal value problem for the logarithmic nonlinearity with Hamiltonians

\[ H(x, p) = (1 + |p|^2)^{\gamma/2} + V(x), \]

with \( 1 < \gamma < \frac{5}{4} \), admits smooth solutions.
The proof in three lines

- $\|u\| \leq C + C\|\ln m\|$
- $\|\ln m\| \leq C + C\|u\|^\beta$
- "result" ($\|u\|$ bounded) follows if $\beta < 1$. 
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The proof in three lines

- $\|u\| \leq C + C \| \ln m \|$
- $\| \ln m \| \leq C + C \| u \|^\beta$
- "result" ($\| u \|$ bounded) follows if $\beta < 1$. 
The key estimate is the following

\[ \frac{d}{dt} \int_{T^d} \frac{1}{m} \leq C \| D_pH \|^2 \| L^\infty \int_{T^d} \frac{1}{m}. \]

This estimate makes it possible to prove a bound

\[ \int_{T^d} | \ln m |^p \leq C + C \| D_pH \|^2 \| L^\infty \|^p. \]
The key estimate is the following

\[ \frac{d}{dt} \int_{\mathbb{T}^d} \frac{1}{m} \leq C \|D_p H\|^2_{L^\infty} \int_{\mathbb{T}^d} \frac{1}{m}. \]

This estimate makes it possible to prove a bound

\[ \int_{\mathbb{T}^d} |\ln m|^p \leq C + C \|D_p H\|^2_{L^\infty}. \]
The short time problem

- $H \sim |p|^\gamma$ for large $p$, and

\[
\begin{aligned}
-u_t + m^\alpha H(x, \frac{Du}{m^\alpha}) &= \Delta u \\
m_t - \text{div}(D_p H(x, \frac{Du}{m^\alpha})m) &= \Delta m.
\end{aligned}
\]

- $u(x, T)$ and $m(x, 0)$ given
- $T$ small.
Some a-priori bounds

Theorem (D. G. - V. Voskanyan)

Suppose $\alpha < \frac{2}{d-2}$ and $\gamma < 2$. If $T$ is small enough then $\frac{1}{m} \in L^r$, for any $r$. 
The proof of the theorem relies on the following estimate:
Assume $\alpha < \frac{2}{d-2}$, $r > 1$, then there exists $q > r$ such that

$$
\int_{\mathbb{T}^d} \frac{1}{m^r(x, t)} \, dx + \int_0^t \int_{\mathbb{T}^d} \left| D \frac{1}{m^{r/2}} \right|^2 \, dx \, dt + \int_0^t \int_{\mathbb{T}^d} \frac{|Du|^\gamma}{m^{r+\alpha}} \, dx \, dt \\
\leq C + C \int_0^t \int_{\mathbb{T}^d} \frac{1}{m^q} \, dx \, dt.
$$
Open questions

- General stationary congestion models
- Time dependent logarithmic non-linearity $\gamma \geq \frac{5}{4}$.
- Long time congestion problem.