On the Capacity of Multiple Access and Broadcast Fading Channels with Full Channel State Information at Low Power Regime

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***Abstract*—We study the throughput capacity region of the Gaussian multi-access (MAC) fading channel with perfect channel state information (CSI) at the receiver and at the transmitters (CSI-TR), at low power regime. We show that it has a multidimensional rectangle structure and thus is simply characterized by single user capacity points. More specifically, we show that at low power regime, the boundary surface of the capacity region shrinks to a single point corresponding to the sum-rate maximizer and that the coordinates of this point coincide with single user capacity bounds. Using the duality of Gaussian MAC and broadcast channels (BC), we provide a simple characterization of the BC capacity region at low power regime.**

***Index Terms*—Multi-access, broadcast, ergodic capacity, capacity region, low-SNR, low power, fading channel, on-off signaling.**

I. Introduction

It is now widely accepted that energy efficiency is a key parameter in designing wireless communication systems. This has catalyzed interests of many researchers inside the information/communication theory communities in order to better understand performance limits of wireless communication in the low power regime, and develop new techniques to achieve/approach these limits, e.g., [1]–[5]. For instance, in wide band communications, although the signal strength is generally very low, one can capitalize on the huge bandwidth and still achieve a high capacity [2], [6], [7]. The low-SNR framework is also useful to model cellular networks in some specific cases [4], [8], sensor networks where power saving is detrimental [9], [10] and more generally any communication scenario where the bandwidth and the power are fixed, but the system degree of freedom is large enough such that the power per degree of freedom is very low [1], [11].

In this paper, we aim at studying the throughput capacity region of multi-access (MAC) and broadcast (BC) fading channels with Gaussian noise, where perfect channel state information (CSI) at the receiver(s) and at the transmitter(s) (CSI-TR) is assumed, at low power regime. The throughput capacity of the Gaussian MAC fading channel has been derived

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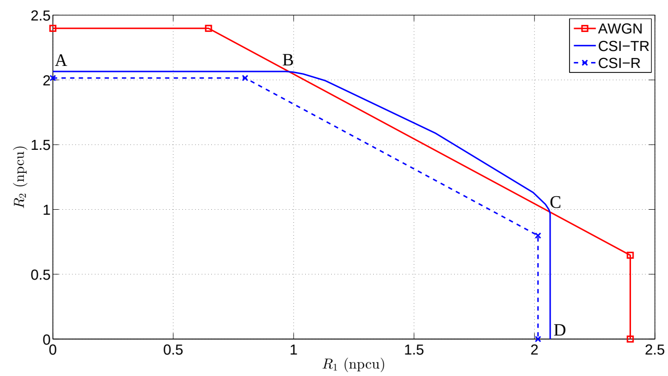

Figure 1. Capacity region of a 2-user MAC Rayleigh fading channel, with $P_1 = P2 = 10$ dB.

in [12]. Therein, it has been shown that each point on the boundary surface of the capacity region can be obtained by successive decoding and that the optimal rate and power allocations can be seen as the generalization of the single-user water-filling construction to MAC channels. The boundary surface is defined as the set of users' rates such that no component can be increase with other components remaining fixed, while remaining in the capacity region [12]. For instance, the capacity region of a 2-user symmetric MAC channel is depicted in Fig. 1. The boundary surface is the curved line joining the points B and C. Differently from the nonfading Gaussian MAC channel where the boundary surface of the capacity region is a line with slope -1, there is no linear part on the boundary surface of the Gaussian MAC fading channels capacity region. Furthermore, since the horizontal line joining A and B and the vertical line joining C and D are completely defined by single user capacity bounds, then determining the boundary surface B-C is enough to fully characterize the MAC capacity region. However, to obtain each point on the boundary surface, a parameterized optimization problem must be solved.

On the other hand, the broadcast channel (BC) is generally a good model for downlink communications in cellular networks where a base station is sending common and/or independent messages to different users. The Single-Input Single-Output (SISO) Gaussian BC is by nature a degraded channel for which the capacity is known [13]. Even with fading along with perfect CSI-TR, the capacity region has been obtained in [14] and in the context of parallel Gaussian channels in [15]. Here again, although the optimal power profile structure is essentially a water filling, an explicit characterization of the boundary capacity region seems difficult to obtain.

We focus in this paper on the low power regime (formally defined later) and analyze the capacity region of the Gaussian MAC fading channel. We show that interestingly, at low power regime, the MAC capacity region has a multidimensional rectangle structure, i.e. the boundary surface reduces to a single point. This point corresponds to the sum-rate maximizer, the components of which are single-user capacity bounds. Using the duality of Gaussian MAC and BC, we provide a simple characterization of the BC capacity region at low power regime.

II. System Model

We consider an uplink scenario where K users are communicating with a base station. All terminals have a single antenna each, i.e., a SISO MAC channel. We focus on a discrete-time Gaussian MAC in which the received signal at the base station at time instant n, $n = 1, \ldots, \infty$, is given by

$$y(n) = \sum_{k=1}^{K} h_k(n)\, x_k(n) + v(n), \tag{1}$$

where $h_k(n)$, $x_k(n)$ and $v(n)$ are complex random variables (r.v.) that represent the channel gain and the transmitted signal of user k and the additive noise, respectively. For convenience, we denote a circularly symmetric complex Gaussian r.v., say r, with mean zero, and variance σ^2, as $r \sim \mathcal{CN}\left(0, \sigma^2\right)$. We assume without loss of generality a normalized Gaussian noise so that $v \sim \mathcal{CN}(0, 1)$. Each user is constrained by an average transmit power $\bar{P}_k$.

We also consider the K-user discrete-time Gaussian BC in which the received signal at user k, $k = 1, \ldots, K$, is given by

$$y_k(n) = h_k(n)\, x(n) + v_k(n), \tag{2}$$

where $x(n)$ and $v_k(n)$ are complex r.v. that represent the transmitted signal and the additive noise at user k. We also assume without loss of generality a normalized Gaussian noise so that $v_k \sim \mathcal{CN}(0, 1)$. The transmitter is constrained by an average transmit power $\bar{P}$.

We focus on fading processes with continuous probability density function (pdf) and with infinite support (although the later assumption is not mandatory). Furthermore, we assume that the channel gains of all users are independent, but not necessarily identically distributed. In addition, in both the MAC and the BC described by (1) and (2), respectively, perfect CSI-TR is assumed, implying that at time instant n, each terminal knows perfectly all channel gains $h_k(n)$, $k = 1, \ldots, K$. For convenience, we let $\gamma_k = |h_k|^2$. We focus on asymptotically low power regime meaning that $\max_{k=1,\ldots,K} \bar{P}_k \to 0$ for the MAC and that $\bar{P} \to 0$ for the BC; and we say that $f(x) \overset{a}{\approx} g(x)$ if and only if $\lim_{x \to a} \frac{f(x)}{g(x)} = 1$. Inequalities $\tilde{<}$ and $\tilde{>}$ are defined analogously. The last definition extends to functions of severable variables where the definition of limits is standard. When it is clear from the context, we omit a in $\overset{a}{\approx}$ for convenience. Finally, bold face letters indicate vectors of dimension K, i.e., $\mathbf{R} = (R_1, \ldots, R_K)$.

III. MAC Capacity Region At Low Power Regime

In this section, we first present our main result in Theorem 1 followed by the proof.[1]

Theorem 1: Let $\mathcal{R}\left(\bar{\mathbf{P}}\right)$ be the multidimensional rectangle defined by the single user capacities, i.e., $\mathcal{R}\left(\bar{\mathbf{P}}\right) = \left\{\mathbf{R} : R_k \le C_k\left(\bar{P}_k\right), \quad k = 1, \ldots, K\right\}$, where $C_k\left(\bar{P}_k\right)$ is the single user capacity of user k, with average transmit power $\bar{P}_k$, and with perfect CSI-TR. For the SISO MAC described by (1), the capacity region $C_{MAC}\left(\bar{\mathbf{P}}\right)$ coincides with $\mathcal{R}\left(\bar{\mathbf{P}}\right)$ at asymptotically low power regime. That is, for any point $\mathbf{R} \in$ on the boundary of the capacity region $C_{MAC}\left(\bar{\mathbf{P}}\right)$, we have:

$$\lim_{\bar{\mathbf{P}} \to \mathbf{0}} \frac{R_k\left(\bar{\mathbf{P}}\right)}{C_k\left(\bar{\mathbf{P}}\right)} = 1, \tag{3}$$

for all $k = 1, \ldots, K$.

Proof: Before proving Theorem 1, we note first that there is no loss of rigor by considering $C_k\left(\bar{\mathbf{P}}\right)$ in (3) although $C_k(\cdot)$ depends only on $\bar{P}_k$. Then, since our focus is on asymptotically low power regime, both $\mathcal{R}\left(\bar{\mathbf{P}}\right)$ and $C_{MAC}\left(\bar{\mathbf{P}}\right)$ collapse to the zero point. However, the characterization of the capacity region adopted in Theorem 1 is in the sense that the limit of the ratio between the achievable rate and the single user capacity is equal to 1 as $\bar{\mathbf{P}} \to 0$ [17]. Next, we want to show that $\forall\ \boldsymbol{\mu} = (\mu_1, \ldots, \mu_K) \in \mathbb{R}_+^K$, the point on the boundary surface of the capacity region, $\mathbf{R}^*(\boldsymbol{\mu})$ is independent of $\boldsymbol{\mu}$, i.e., all surfaces parameterizing the boundary surface of the capacity region intersect in exactly one point. This direct approach seems a bit complicated since the expression of $\boldsymbol{R_k^*}$ in [12, Theorem 3.16] is somehow complicated. Instead, we adopt an information-theoretical approach to prove Theorem 1. Clearly, the region $\mathcal{R}\left(\bar{\mathbf{P}}\right)$ is an upper bound on the MAC capacity region (for arbitrary $\bar{\mathbf{P}}$, not necessarily low) since the former is only constrained by single user capacities of all users. To show that all points in this set are achievable, it suffices to sow that the point $\mathbf{C}\left(\bar{\mathbf{P}}\right) = \left(C_1\left(\bar{P}_1\right), \ldots, C_K\left(\bar{P}_K\right)\right)$ is asymptotically achievable. For this purpose, let us compute $\mathbf{R}^*(\boldsymbol{\mu})$ for $\mu_1 = \mu_2 = \ldots = \mu_K = 1$, i.e., the point on the boundary of the capacity region that maximizes the sum-rate. The kth component, $k = 1, \ldots, K$, of this point is given by [12]:

$$\boldsymbol{R_k^*} = \int_{\lambda_k}^{\infty} \log\left(\frac{\gamma}{\lambda_k}\right) \prod_{i \neq k} F_i\left(\frac{\lambda_i}{\lambda_k}\gamma\right) f_k(\gamma)\, \mathrm{d}\gamma, \tag{4}$$

[1] Theorem 1 refines the statement of Theorem 1 in [16], although in essence, both results are similar.

where $f_k(\cdot)$ and $F_k(\cdot)$ denote the pdf of the channel gain power γ_k and its cumulative distribution function (cdf), respectively; and where the constants λ_k's satisfy

$$\bar{P}_k = \int_{\lambda_k}^{\infty} \left(\frac{1}{\lambda_k} - \frac{1}{\gamma}\right) \prod_{i \neq k} F_i\left(\frac{\lambda_i}{\lambda_k}\gamma\right) f_k(\gamma)\,\mathrm{d}\gamma, \tag{5}$$

for $k = 1, \ldots, K$. Let us define a function $G_k(x_1, \ldots, x_K)$, as the RHS of (5), i.e.,

$$G_k(x_1, \ldots, x_K) = \int_{x_k}^{\infty} \left(\frac{1}{x_k} - \frac{1}{\gamma}\right) \prod_{i \neq k} F_i\left(\frac{x_i}{x_k}\gamma\right) f_k(\gamma)\,\mathrm{d}\gamma, \tag{6}$$

for positive $x_1, \ldots, x_K$. Note that $G_k(\lambda_1, \ldots, \lambda_K) = \bar{P}_k$ for all $k = 1, \ldots, K$. Because each average power constraint depends on all λ_k's, it is natural to consider that each λ_k is a function of all $\bar{P}_k$'s, i.e., $\lambda_k = \lambda_k\left(\bar{P}_1, \ldots, \bar{P}_K\right)$.[2] We then claim the result in the following lemma.

Lemma 1: For the λ_k's that satisfy the average power constraint (5), it holds that:

$$\lim_{\bar{\mathbf{P}} \to 0} \lambda_k\left(\bar{P}_1, \ldots, \bar{P}_K\right) = \infty \tag{7}$$

for all $k = 1, \ldots, K$. where $\bar{\mathbf{P}} \to \mathbf{0}$ stands for $\left(\bar{P}_1, \ldots, \bar{P}_K\right) \to (0, \ldots, 0)$.

Proof: The intuition behind (7) is that at low power regime, λ_k's which have an interpretation of the power price converge toward infinity. This is true because the fading channels considered have infinite support. Details of the proof are omitted here due to length constraint. ■

Now, since $\forall \gamma \in [\lambda_k, \infty)$, we have $F_i(\lambda_i) \leq F_i\left(\frac{\lambda_i}{\lambda_k}\gamma\right) \leq 1$, the following inequalities hold true for all $\lambda_k > 0$:

$$\begin{aligned}
\prod_{i \neq k} F_i(\lambda_i) \int_{\lambda_k}^{\infty} \left(\frac{1}{\lambda_k} - \frac{1}{\gamma}\right) f_k(\gamma)\,\mathrm{d}\gamma \leq \\
\int_{\lambda_k}^{\infty} \left(\frac{1}{\lambda_k} - \frac{1}{\gamma}\right) \prod_{i \neq k} F_i\left(\frac{\lambda_i}{\lambda_k}\gamma\right) f_k(\gamma)\,\mathrm{d}\gamma \leq \\
\int_{\lambda_k}^{\infty} \left(\frac{1}{\lambda_k} - \frac{1}{\gamma}\right) f_k(\gamma)\,\mathrm{d}\gamma,
\end{aligned} \tag{8}$$

or equivalently,

$$\prod_{i \neq k} F_i(\lambda_i) \leq \frac{\int_{\lambda_k}^{\infty} \left(\frac{1}{\lambda_k} - \frac{1}{\gamma}\right) \prod_{i \neq k} F_i\left(\frac{\lambda_i}{\lambda_k}\gamma\right) f_k(\gamma)\,\mathrm{d}\gamma}{\int_{\lambda_k}^{\infty} \left(\frac{1}{\lambda_k} - \frac{1}{\gamma}\right) f_k(\gamma)\,\mathrm{d}\gamma} \leq 1. \tag{9}$$

Taking the limits as $\bar{\mathbf{P}} \to \mathbf{0}$ on both sides of (9) and using Lemma 1, we establish that

$$\lim_{\bar{\mathbf{P}} \to 0} \frac{\int_{\lambda_k}^{\infty} \left(\frac{1}{\lambda_k} - \frac{1}{\gamma}\right) \prod_{i \neq k} F_i\left(\frac{\lambda_i}{\lambda_k}\gamma\right) f_k(\gamma)\,\mathrm{d}\gamma}{\int_{\lambda_k}^{\infty} \left(\frac{1}{\lambda_k} - \frac{1}{\gamma}\right) f_k(\gamma)\,\mathrm{d}\gamma} = 1. \tag{10}$$

[2]Although rigorously speaking, in the context of the MAC, for any $k = 1, \ldots, K$, $\lambda_k(\cdot)$ is a multivariate function of all $\bar{P}_i$, $i = 1, \ldots, K$, i.e., $\lambda_k = \lambda_k\left(\bar{P}_1, \ldots, \bar{P}_K\right)$, we will find it convenient to denote it simply as λ_k when there is no ambiguity.

Combining (5) and (10), we have:

$$\bar{P}_k \approx \int_{\lambda_k}^{\infty} \left(\frac{1}{\lambda_k} - \frac{1}{\gamma}\right) f_k(\gamma)\,\mathrm{d}\gamma. \tag{11}$$

Along similar lines, one can also prove that

$$\lim_{\bar{\mathbf{P}} \to 0} \frac{\int_{\lambda_k}^{\infty} \log\left(\frac{\gamma}{\lambda_k}\right) \prod_{i \neq k} F_i\left(\frac{\lambda_i}{\lambda_k}\gamma\right) f_k(\gamma)\,\mathrm{d}\gamma}{\int_{\lambda_k}^{\infty} \log\left(\frac{\gamma}{\lambda_k}\right) f_k(\gamma)\,\mathrm{d}\gamma} = 1. \tag{12}$$

But due to (11), the RHS of (12) is exactly the capacity of single user k over the fading γ_k and subject to power constraint $\bar{P}_k$. Hence, we conclude that at asymptotically low power regime, $R_k^*\left(\bar{P}_k\right) \approx C_k\left(\bar{P}_k\right)$, for all $k = 1, \ldots, K$. Finally, we highlight the fact that while $\lambda_k = \lambda_k\left(\bar{P}_1, \ldots, \bar{P}_K\right)$ for arbitrary $\bar{P}_k$'s values, (11) stipulates that at asymptotically low power regime, the dependence on $\bar{P}_i$'s, $i \neq k$, breaks down so that $\lambda_k = \lambda_k\left(\bar{P}_k\right)$ only. Furthermore, both (10) and (11) infer that the power price in single user or multiple access channels is the same at asymptotically low power regime. ■

It is worthwhile to mention that in order for our result to hold, we only require the fading of different users to be independent, not necessarily identically distributed. We also note that Theorem 1 emphasizes on how communication at low power regime can be energy efficient: at this regime, each user benefits from a single user performance as if others do not exist. The strategy to achieve this is the same as the one that maximizes the sum-rate, a time-division multiple-access (TDMA) strategy where at most one user is allowed to transmit at any given state. Since in the MAC scenario we perform water-filling over the maximum of users channel gains $\max_{k=1,\ldots,K} \gamma_k$ to achieve the maximum sum-rate, whereas in the single-user case, we perform waterfilling on γ_k, for each individual user, then these rates seem to be different a priori. However, the two strategies perform similarly at asymptotically low power regime as suggested by Theorem 1. At this low power regime, if each user transmits only when its channel gain is extremely good, then it is more unlikely that other users channels be better. The proof of Theorem 1 is somehow too technical and to get the feel of the insight behind Theorem 1, we consider the following simple example.

Example 1: Let us consider a symmetric MAC channel where all user have the same fading statistics and are constrained by the same single average power constraint $\bar{P}_k = \bar{P}$, for $k = 1, \ldots, K$. As discussed above, TDMA achieves the maximum sum-rate. This yields a sum-rate equal to $R_\Sigma = \mathop{\mathbf{E}}_{\gamma_{max}}\left[\log\left(1 + \gamma_{max} P\left(\gamma_{max}\right)\right)\right]$, where $\gamma_{max} = \max_{k=1,\ldots,K} \gamma_k$, where $P(x) = \left[\frac{1}{\lambda} - \frac{1}{x}\right]^+$ and where λ is chosen such that $\mathop{\mathbf{E}}_{\gamma_{max}}\left[P\left(\gamma_{max}\right)\right] = K\bar{P}$ and the factor K is due to the fact that each user has probability $1/K$ (symmetric fading) of being the one that has maximum channel gain. By symmetry, each user gets the rate $R_k = \frac{1}{K} \mathop{\mathbf{E}}_{\gamma_{max}}\left[\log\left(1 + \gamma_{max} P\left(\gamma_{max}\right)\right)\right]$ which can be lower-bounded as follows:

$$R_k = \mathop{\mathbf{E}}_{\gamma_k}\left[\log\left(1 + \gamma_k P\left(\gamma_k\right)\right) \mathrm{Prob}\left\{\gamma_j \leq \gamma_k,\ j \neq k\right\}\right] \tag{13}$$

$$\geq \quad \mathop{\mathbf{E}}_{\gamma_k}\left[\log\left(1+\gamma_k P\left(\gamma_k\right)\right) \operatorname{Prob}\left\{\gamma_j \leq \lambda,\ j \neq k\right\}\right] \quad (14)$$

$$= \quad \operatorname{Prob}\left\{\gamma_j \leq \lambda,\ j \neq k\right\} \mathop{\mathbf{E}}_{\gamma_k}\left[\log\left(1+\gamma_k P\left(\gamma_k\right)\right)\right] \quad (15)$$

where (14) follows because only $\gamma_k \geq \lambda$ matters in (13). Since $\lambda \to \infty$ as $\bar{P} \to 0$, then using (15), we have:

$$\lim_{\bar{P}\to 0} \frac{R_k}{\mathop{\mathbf{E}}_{\gamma_k}\left[\log\left(1+\gamma_k P\left(\gamma_k\right)\right)\right]} \geq 1. \quad (16)$$

From (13), we also have:

$$\frac{R_k}{\mathop{\mathbf{E}}_{\gamma_k}\left[\log\left(1+\gamma_k P\left(\gamma_k\right)\right)\right]} \leq 1. \quad (17)$$

Combining (16) and (17), we obtain:

$$\lim_{\bar{P}\to 0} \frac{R_k}{\mathop{\mathbf{E}}_{\gamma_k}\left[\log\left(1+\gamma_k P\left(\gamma_k\right)\right)\right]} = 1. \quad (18)$$

Along similar steps, one can also show that:

$$\lim_{\bar{P}\to 0} \frac{\bar{P}}{\mathop{\mathbf{E}}_{\gamma_k}\left[P\left(\gamma_k\right)\right]} = 1. \quad (19)$$

Note that (19) asserts that λ satisfies the single user power constraint asymptotically at low power regime, i.e., $\mathop{\mathbf{E}}_{\gamma_k}\left[P\left(\gamma_k\right)\right] \approx \bar{P}$; hence $C_k\left(\bar{P}\right) \approx \mathop{\mathbf{E}}_{\gamma_k}\left[\log\left(1+\gamma_k P\left(\gamma_k\right)\right)\right]$, which combined with (18) yields $R_k \approx C_k\left(\bar{P}\right)$ as predicted by Theorem 1.

IV. Capacity Region of the BC At Low Power Regime

In order to characterize the BC capacity region at low power regime, we utilize the established duality between the Gaussian MAC and BC so that we can deduce the BC capacity region from that of the MAC given in Theorem 1 [18]. Recall that the duality of these channels implies that the capacity region of the BC with power $\bar{P}$, is exactly equal to the capacity of the dual MAC, which has the same channel gains, and a sum power constraint of $\bar{P}$ across all K transmitters. That is $C_{BC}\left(\bar{P}\right) = \bigcup_{\mathbf{1}\cdot\boldsymbol{P}=\bar{P}} C_{MAC}\left(\boldsymbol{P}\right)$, where $\boldsymbol{\alpha}\cdot\boldsymbol{P} = \sum_{k=1}^{K}\alpha_k P_k$ [18]. Our result is formalized in Theorem 2.

Theorem 2: Let $\mathcal{R}'\left(\bar{P}\right)$ be the region defined by $\mathcal{R}'\left(\bar{P}\right) = \left\{\mathbf{R} : R_k \leq C_k\left(\alpha_k \bar{P}\right),\ k = 1,\ldots,K,\ \sum_{k=1}^{K}\alpha_k = 1\right\}$, where $C_k\left(\alpha_k \bar{P}\right)$ is the single user capacity of user k, with average transmit power $\alpha_k \bar{P}$, and with perfect CSI-TR; and where α_k's are arbitrary positive coefficients. For the SISO BC described by (2), the capacity region $C_{BC}\left(\bar{P}\right)$ coincides with $\mathcal{R}'\left(\bar{P}\right)$ at asymptotically low power regime. That is, all points in $\mathcal{R}'\left(\bar{P}\right)$ are achievable and vice versa any point in $C_{BC}\left(\bar{P}\right)$ is necessarily in $\mathcal{R}'\left(\bar{P}\right)$ too.[3]

Proof: To show that any point in $\mathcal{R}'\left(\bar{P}\right)$ is achievable, we need only to focus on rates on the boundary of $\mathcal{R}'\left(\bar{P}\right)$, i.e., the rates such that $R_k = C_k\left(\alpha_k \bar{P}\right)$, for all $k = 1,\ldots,K$. Then, we know from Theorem 1 that the point with coordinates $\left(C_1\left(\alpha_1 \bar{P}\right),\ldots,C_K\left(\alpha_K \bar{P}\right)\right)$ belongs to $C_{MAC}\left(\bar{P}\boldsymbol{\alpha}\right)$. Since $\sum_{k=1}^{K}\alpha_k \bar{P} = \bar{P}$, then the point $\left(C_1\left(\alpha_1 \bar{P}\right),\ldots,C_K\left(\alpha_K \bar{P}\right)\right)$ also belongs to $C_{BC}\left(\bar{P}\right)$ by the duality between the Gaussian MAC and BC. This completes the achievability part.

To prove the converse, we only need to show that the points on the boundary of the BC capacity region necessarily belong to $\mathcal{R}'\left(\bar{P}\right)$ too. To that end, let $\boldsymbol{R}$ be a point on the boundary of the BC capacity region. Then again, by the duality between the MAC and the BC, there exists a power policy $\boldsymbol{P}$ such $\bar{P} = \sum_{k=1}^{K}P_k$ and $\boldsymbol{R} \in C_{MAC}\left(\boldsymbol{P}\right)$. For convenience, we let $P_k = \alpha_k \bar{P}$, for some positive α_k's, so that we have $\sum_{k=1}^{K}\alpha_k = 1$. Furthermore, since $\boldsymbol{R}$ is on the boundary of the BC capacity region, then it is necessarily on the boundary of the $C_{MAC}\left(\boldsymbol{P}\right)$ too, otherwise this would contradict the definition of the boundary curve. But, from Theorem 1 we know that the boundary of the MAC capacity region shrinks to a single point at low power regime, corresponding to the sum-rate maximizer and that $R_k\left(P_k\right) \approx C_k\left(P_k\right)$, or equivalently $R_k\left(\alpha_k\bar{P}\right) \approx C_k\left(\alpha_k\bar{P}\right)$. Therefore, the converse is also true and this completes the proof of Theorem 2. ■

Again, in order for Theorem 2 to hold, the fading of different users need not be identically distributed, only the independence is instrumental. We also note that from the proof above, since each point on the BC boundary region corresponds to a sum-rate maximizer of a certain dual MAC, then to achieve a given point on the boundary of the BC capacity region, an optimal strategy is to use the MAC-BC transformation in each fading state to find a BC power policy that achieves the same average rate as the corresponding dual MAC [18].

V. Numerical Results and Discussion

We first present numerical results for a 2-user MAC channel where both users undergo independent Rayleigh fading channels and where users' powers are set to $\bar{P}_1 = -30$ dB and $\bar{P}_2 = -40$ dB. Figure 2 depicts the capacity region in nats per channel use (npcu) obtained numerically using the characterization in [12, Theorem 3.16] along with the capacity regions of an additive white Gaussian noise (AWGN) MAC channel and that of a Rayleigh fading MAC channel with CSI-R only. As it can be seen in Fig. 2, the throughput capacity region coincides with a rectangle as predicted by Theorem 1. For the BC, Fig. 3 and Fig. 4 display the capacity region obtained numerically, the one given by Theorem 2 along with time-sharing achievable region, for $\bar{P} = -10$ dB and $\bar{P} = -70$ dB, respectively. The AWGN BC capacity region and that of Rayleigh fading BC with CSI-R only are also displayed to highlight the tremendous gain due to CSI-T. Recall that while the capacity region of the fading BC with CSI-R only is generally not know, it is actually know in the symmetric fading case considered here. We note in both figures the accuracy of the characterization in Theorem 2. Although time-sharing

[3] Here again, as mentioned previously at the beginning of the proof of Theorem 1, the achievability and the converse are in the sense that the limit of the ratio is equal to 1 as $\bar{P} \to 0$.

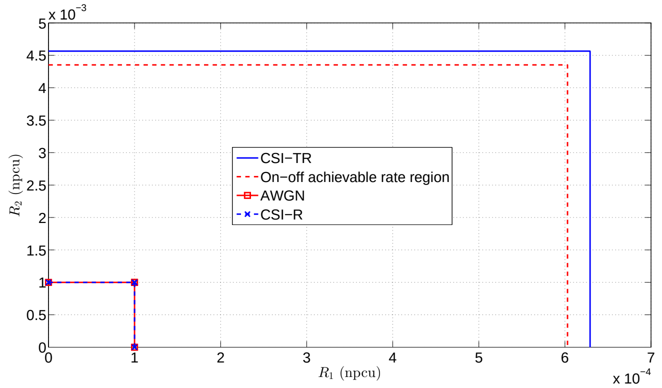

Figure 2. Capacity region of a 2-user MAC Rayleigh fading channel, with $\bar{P}_1 = -40$ dB and $\bar{P}_2 = -30$ dB.

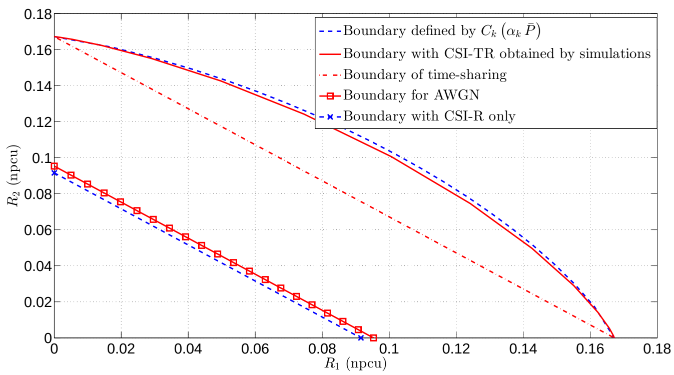

Figure 3. Capacity region of a 2-user BC Rayleigh fading channel, with $\bar{P} = -10$ dB.

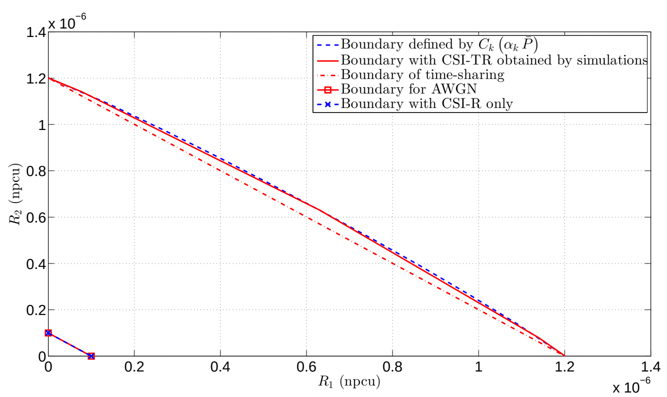

Figure 4. Capacity region of a 2-user BC Rayleigh fading channel, with $\bar{P} = -70$ dB.

achievable region is strictly suboptimal at −10 dB, it converges slowly to the capacity region as shown in Fig. 4 in the case of Rayleigh BC.

VI. Conclusion

We have analyzed the throughput capacity region of the MAC fading channel with perfect CSI-TR at low power regime. While the capacity region has a polymatroid structure at an arbitrary power regime, we have shown that it is simply a multidimensional rectangle at asymptotically low power regime and that each user can achieve a single user performance as if others do not exist. We have also deduced the BC capacity region using the duality and provided a simple asymptotic characterization of the BC capacity region.

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